SEVENTH WEEK HOMEWORK ASSIGNMENT

Problem 1. Prove that if $E \subset \mathbb{R}^d$ is Lebesgue measurable, then for any subset S of $\mathbb{R}^d$ we have

$$m^*(S) = m^*(S \cap E) + m^*(S \setminus E).$$

Hint: We have already established this property in class for *bounded* sets. You only need to use the "larger and larger" boxes trick to extend this property to unbounded sets.

Problem 2. Prove that if s and σ are simple functions on $\mathbb{R}^d$, and if $c \in \mathbb{R}$, then $s + \sigma$, $s - \sigma$, $s \cdot \sigma$ and $c\,s$ are also simple functions.

Hint: The only tricky part is $s \cdot \sigma$. Use (after convincing yourselves that it is true) the following simple property of indicator functions:

$$\mathbf{1}_E \cdot \mathbf{1}_F = \mathbf{1}_{E \cap F}.$$

Problem 3. Prove that if s and σ are nonnegative simple functions on $\mathbb{R}^d$ and if $c \in [0, \infty]$, then

$$\int_{R^d} (s + \sigma)(x)\,dx = \int_{\mathbb{R}^d} s(x)\,dx + \int_{\mathbb{R}^d} \sigma(x)\,dx.$$

$$\int_{\mathbb{R}^d} c\,s(x)\,dx = c\int_{\mathbb{R}^d} s(x)\,dx.$$