SIXTH WEEK HOMEWORK ASSIGNMENT

Problem 1. Prove that if $E_1, E_2, \ldots$ is a countable sequence of Lebesgue measurable sets, then $\bigcap_{n=1}^{\infty} E_n$ is also Lebesgue measurable.

Hint: Use one of de Morgan's rules and the analogue result proven in class for countable unions of measurable sets.

Problem 2. Show that for any sets $F \subset E \subset \mathbb{R}^d$ we have

$$m^*(E) - m^*(F) \leq m^*(E \setminus F).$$

Hint: Use the sub-additivity of the Lebesgue outer measure.

Problem 3. Let $E \subset \mathbb{R}^d$ be a any set. Show that there is a Lebesgue *measurable* set E' such that $E \subset E'$ and $m^*(E) = m(E')$.

Hint: Use the outer regularity of the Lebesgue outer measure (approximation by open sets from without).

The goal of the next problem is to prove several *equivalent* ways of describing Lebesgue measurability.

Problem 4. Let $E \subset \mathbb{R}^d$. Prove that the following are equivalent:

(1) E is Lebesgue measurable, meaning: for all $\epsilon > 0$ there is U open such that $E \subset U$ and $m^*(U \setminus E) < \epsilon$.
(2) E is "almost open" in the following sense: for all $\epsilon > 0$ there is U open such that $m^*(E \triangle U) < \epsilon$.
(3) E has a good "inner approximation by closed sets": for all $\epsilon > 0$ there is F closed such that $F \subset E$ and $m^*(E \setminus F) < \epsilon$.
(4) E is "almost closed" in the following sense: for all $\epsilon > 0$ there is F closed such that $m^*(E \triangle F) < \epsilon$.

Hint: Similar to the analogue statements for Jordan measurable sets.

Problem 5. Prove the following inner regularity property of the Lebesgue measure. If $E \subset \mathbb{R}^d$ is Lebesgue measurable, then

$$m(E) = \sup\{m(K) \colon K \text{ is compact, } K \subset E\}.$$

Hint: First assume that E is bounded and use the "inner approximation by closed sets" characterization of measurable sets.

If E is unbounded, consider its intersection with larger and larger boxes and use the monotone convergence theorem for sets.

The goal of the next problem is to characterize in different (equivalent) ways Lebesgue measurable sets of *finite* measure.

Problem 6. Let $E \subset \mathbb{R}^d$. Prove that the following are equivalent:

(1) E is Lebesgue measurable and $m(E) < \infty$.

(2) "approximation by open with finite measure": for all $\epsilon > 0$ there is U open with $m(U) < \infty$ such that $E \subset U$ and $m^*(U \setminus E) < \epsilon$.

(3) "almost open and bounded": for all $\epsilon > 0$ there is U open and bounded such that $m^*(E \triangle U) < \epsilon$.

(4) "inner approximation by compact": for all $\epsilon > 0$ there is K compact such that $K \subset E$ and $m^*(E \setminus K) < \epsilon$.

(5) "almost compact": for all $\epsilon > 0$ there is K compact such that $m^*(E \triangle K) < \epsilon$.

(6) "almost measurable and bounded": for all $\epsilon > 0$ there is S measurable and bounded such that $m^*(E \triangle S) < \epsilon$.

(7) "almost finite measure": for all $\epsilon > 0$ there is S measurable with $m(S) < \infty$ such that $m^*(E \triangle S) < \epsilon$.