FIFTH WEEK HOMEWORK ASSIGNMENT

Problem 1. Prove the translation invariance of the Lebesgue outer measure. That is, show that if $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ then

$$m^*(x + E) = m^*(E).$$

Problem 2. Prove a similar property for multiplication on $\mathbb{R}$: if $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, if we denote $x\, E := \{x \cdot a \colon a \in E\}$, then

$$m^*(x\, E) = \left|x\right| m^*(E).$$

Problem 3. Let $E \subset \mathbb{R}$ with $m^*(E) < \infty$.
Define the function $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := m^*(E \cap (-\infty, x]).$$

Prove that f is uniformly continuous on $\mathbb{R}$.

Problem 4. Let $f \colon \mathbb{R} \to \mathbb{R}$ be differentiable at all points.

(a) Prove that if for all $x \in \mathbb{R}$ we have $\left|f'(x)\right| \leq 1$, then for every subset $E \subset \mathbb{R}$ we have

$$m^*(f(E)) \leq m^*(E).$$

We say that the function f *contracts* the measure.
Hint: Use the mean value theorem from calculus.

(b) Find an example of a differentiable function f such that for some x, $\left|f'(x)\right| > 1$, and which does not contract the measure.
Hint: Use Problem 2 above.

Problem 5. Find an example of a set $E \subset \mathbb{R}$ for which

$$m^*(E) > \sup \{m^*(U) \colon U \subset E, U \text{ is open}\}.$$

This will show that the *exact* inner regularity analogue of the outer regularity is false.

Problem 6. Prove that if $B_1, B_2, \ldots, B_N$ are *almost* disjoint boxes (meaning that their interiors are disjoint) then

$$m\,(\bigcup_{k=1}^{N} B_k) = \sum_{k=1}^{N} \left|B_k\right|.$$

The measure m above refers to the Jordan measure (since a union of boxes is elementary), which of course coincides with the Lebesgue outer measure.