FIFTH WEEK HOMEWORK ASSIGNMENT

Problem 1. Prove the translation invariance of the Lebesgue outer measure. That is, show that if $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ then

$$m^*(x + E) = m^*(E).$$

Problem 2. Prove a similar property for multiplication on $\mathbb{R}$: if $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, if we denote $x\, E := \{x \cdot a \colon a \in E\}$, then

$$m^*(x\, E) = \left|x\right| m^*(E).$$

Problem 3. Let $E \subset \mathbb{R}$ with $m^*(E) < \infty$.
Define the function $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := m^*(E \cap (-\infty, x]).$$

Prove that f is uniformly continuous on $\mathbb{R}$.

Problem 4. Let $f \colon \mathbb{R} \to \mathbb{R}$ be differentiable at all points.

(a) Prove that if for all $x \in \mathbb{R}$ we have $\left|f'(x)\right| \leq 1$, then for every subset $E \subset \mathbb{R}$ we have

$$m^*(f(E)) \leq m^*(E).$$

We say that the function f *contracts* the measure.
Hint: Use the mean value theorem from calculus.

(b) Find an example of a differentiable function f such that for some x, $\left|f'(x)\right| > 1$, and which does not contract the measure.
Hint: Use Problem 2 above.

Problem 5. Find an example of a set $E \subset \mathbb{R}$ for which

$$m^*(E) > \sup \{m^*(U) \colon U \subset E, U \text{ is open}\}.$$

This will show that the *exact* inner regularity analogue of the outer regularity is false.

Problem 6. Prove that if $B_1, B_2, \ldots, B_N$ are *almost* disjoint boxes (meaning that their interiors are disjoint) then

$$m\,(\bigcup_{k=1}^{N} B_k) = \sum_{k=1}^{N} \left|B_k\right|.$$

The measure m above refers to the Jordan measure (since a union of boxes is elementary), which of course coincides with the Lebesgue outer measure.

It was mentioned in class that the *additivity* (finite of countable) of the Lebesgue outer measure does *not* always hold. However, we can show that it does hold in many interesting cases.

Let us first remember the following useful result.

Lemma. If $E = \bigcup_{n=1}^{\infty} B_n$, where B_n are almost disjoint boxes, then

$$m^*(E) = \sum_{n=1}^{\infty} \left| B_n \right| .$$

Moreover, any *open* set can be expressed in this way, as a countable union of almost disjoint boxes.

We are now ready for the first result on the additivity of the Lebesgue outer measure.

Problem 7. Prove that if U and V are *open* sets and if $U \cap V = \emptyset$, then

$$m^*(U \cup V) = m^*(U) + m^*(V).$$

Hint: Use the lemma above for both U and V.

The next result on the additivity of the Lebesgue outer measure is for compact sets. They require a bit more preparation. We begin with a definition.

Definition. If A and B are two subsets of $\mathbb{R}$ (or of $\mathbb{R}^d$), we define the *distance* between them as the infimal distance between a point in A and a point in B. More precisely,

$$\text{dist}(A, B) := \inf \left\{ \left| x - y \right| : x \in A, y \in B \right\}.$$

Problem 8. Prove that if K and L are compact sets in $\mathbb{R}$ (or in $\mathbb{R}^d$), and if $K \cap L = \emptyset$, then

$$\text{dist}(K, L) > 0 .$$

Hint: Assume by contradiction that $\text{dist}(K, L) = 0$.
At some point you may also need to use the fact that any sequence in a compact set has a convergent subsequence.

Problem 9. Prove that if A and B are two subsets of $\mathbb{R}$ (or of $\mathbb{R}^d$) such that $\text{dist}(A, B) > 0$, then there are two *open* sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Hint: "Thicken" (like we did before in class) each of the sets A and B by a small enough amount - how small will depend on $\text{dist}(A, B)$.

Problem 10. Prove that if K and L are compact sets in $\mathbb{R}$ (or in $\mathbb{R}^d$), and if $K \cap L = \emptyset$, then

$$m^*(K \cup L) = m^*(K) + m^*(L)\,.$$

Hint: Use problem 8, then 9 and then 7.
Also use the fact (proven in class) that for any set E,

$$m^*(E) = \inf\{m^*(U) \colon E \subset U \text{ and } U \text{ is open}\}.$$