ADDITIVITY OF MEASURE FOR COMPACT SETS

This is a sketch of the proof of the additivity of the Lebesgue outer measure for compact sets (Problem 10 from the 5th week assignment).

Let us remember the statement: if K and L are compact sets in $\mathbb{R}$ (or in $\mathbb{R}^d$), and if $K \cap L = \emptyset$, then

$$m^*(K \cup L) = m^*(K) + m^*(L). \tag{1}$$

We will use problems 7, 8 and 9 from the homework as well as the outer regularity of the Lebesgue outer measure.

We begin with the observation that by the sub-additivity of the Lebesgue outer measure, we always have (regardless of compactness) that

$$m^*(K \cup L) \leq m^*(K) + m^*(L).$$

Therefore, in order to establish (1), it is enough to prove the following inequality:

$$m^*(K \cup L) \geq m^*(K) + m^*(L). \tag{2}$$

Let us create an epsilon of room first, and prove that given any $\epsilon > 0$, we have

$$m^*(K \cup L) + \epsilon \geq m^*(K) + m^*(L). \tag{3}$$

If we manage to prove (3), then by letting $\epsilon \to 0$ we derive (2) and we are done.

The regularity of the Lebesgue outer measure applied to $K \cup L$ implies the following: there is an open set D such that $K \cup L \subset D$ and

$$m^*(D) \leq m^*(K \cup L) + \epsilon. \tag{4}$$

Using Problem 8, since K and L are compact and $K \cap L = \emptyset$, we have that $\text{dist}(K, L) > 0$. We may then apply Problem 9 to the sets K and L in order to "separate" them by open sets: there are open sets U and V such that $K \subset U$, $L \subset V$ and $U \cap V = \emptyset$.

The set D is "good" from a measure point of view: it is open and it approximates well enough $K \cup L$.

The sets U and V are good because they are open and they separate K from L.

Let us combine these advantages and define

$$U' := U \cap D \quad \text{and} \quad V' := V \cap D.$$

Note that we still have:

$$U' \text{ and } V' \text{ are open, } U' \cap V' = \emptyset,\ K \subset U',\ L \subset V'.$$

Using Problem 7 for U' and V' and then the monotonicity of the Lebesgue outer measure, we have:

$$m^*(U' \cup V') = m^*(U') + m^*(V') \geq m^*(K) + m^*(L). \tag{5}$$

Moreover, $U' \cup V' \subset D$, so by the monotonicity of the Lebesgue outer measure we have:

$$m^*(U' \cup V') \leq m^*(D). \tag{6}$$

Combining (5) and (6) we obtain:

$$m^*(K) + m^*(L) \leq m^*(D). \tag{7}$$

Combining (4) and (7) we obtain (3), and this completes the proof. □