FOURTH WEEK HOMEWORK ASSIGNMENT

Problem 1. Show that a countable union of Jordan measurable sets may not be Jordan measurable.

Then show that a countable intersection of Jordan measurable sets may not be Jordan measurable.

Hint: Look for an example of a set that is not Jordan measurable, but which you can break down into countably many Jordan measurable subsets. For the second problem, on intersection, consider the complements of these subsets relative to some large enough box.

Problem 2. Finish up the proof of the theorem discussed in class saying that if a sequence $\{f_n\}_{n\geq 1}$ of Riemann integrable functions on $[a, b]$ converges *uniformly* to f, then f is Riemann integrable as well and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Problem 3. Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of $\mathbb{R}$ such that $\mathcal{X} \subset \mathcal{Y}$. Prove that

$$\inf \mathcal{X} \geq \inf \mathcal{Y} \quad \text{ and } \sup \mathcal{X} \leq \sup \mathcal{Y}.$$

Problem 4. Prove that the Jordan outer measure of a set $E \subset \mathbb{R}^d$ is the infimal cost required to cover E by a finite union of boxes.

More precisely, prove that

$$m^{*,J}(E) = \inf \Big\{ \sum_{i=1}^{k} \big|B_i\big| \;:\; E \subset B_1 \cup B_2 \ldots \cup B_k, \text{ where } B_1, B_2, \ldots, B_k \text{ are boxes and } k \geq 1 \Big\}.$$

Note: Finite unions of boxes are elementary sets. So the only difference between what is written above and the definition of the Jordan outer measure is that in the formula above we do *not* require the boxes that cover E to be *disjoint*. In other words, what the problem is saying is that it does not matter if we throw in all boxes that cover E; whether they are disjoint or not, when we take the infimum we get the same quantity.

Problem 5. Prove that a bounded set E is Jordan measurable if and only if it is "almost" elementary, in the sense that: for any $\epsilon > 0$ there is an elementary set B such that $E \subset B$ and $m^{*,J}(B \setminus E) < \epsilon$.

Problem 6. Find a function $f\colon [0,1] \to \mathbb{R}$ that has an *uncountable* set of discontinuities, but which is still Riemann integrable.

Hint: Use Cantor's set.

Problem 7. Prove that if a set $E \subset \mathbb{R}^d$ has Lebesgue *outer* measure zero (we may call such sets *negligible*), then E is Lebesgue measurable.

Problem 8. Prove that the Lebesgue outer measure is monotone: if $E \subset F \subset \mathbb{R}^d$, then $m^*(E) \leq m^*(F)$.

Conclude, using the previous problem, that any subset of a negligible set is Lebesgue measurable and it has Lebesgue measure zero.

Problem 9. Let E be a bounded set in $\mathbb{R}^d$.

(a) Prove that if E is Jordan measurable, then it is Lebesgue measurable as well.

(b) [*] You may try to prove that in this case, the Jordan measure and the Lebesgue measure of E are the same (this is not easy, and we will do it in class anyway).

Hint (for part (a)): Use the fact discussed in class, saying that if we pay just ϵ more, we can always consider *open* boxes (rather than just any boxes) to cover our sets E.