THIRD WEEK HOMEWORK ASSIGNMENT

Problem 1. Prove that the elementary measure is monotone and subadditive. In other words, show that the map $m\colon \mathcal{E}(\mathbb{R}^d) \to \mathbb{R}$ satisfies the following:

(i) For all elementary sets E and F, if $E \subset F$ then $m(E) \leq m(F)$.
(ii) For all elementary sets E and F, $m(E \cup F) \leq m(E) + m(F)$.

Explain why the same properties hold for the Jordan measure.

Problem 2. Prove that the Cantor set is Jordan measurable and that its Jordan measure is 0.

Problem 3. Prove that if E and F are Jordan measurable sets, then $E \cap F$, $E \setminus F$ and $E \triangle F$ are Jordan measurable as well.

Problem 4. Prove that a bounded set $E \subset \mathbb{R}^d$ is Jordan measurable if and only if for all $\epsilon > 0$ there is an elementary set A such that $m^{*,J}(E \triangle A) < \epsilon$.

Problem 5. Let $E \subset \mathbb{R}^d$ be any bounded set. Prove the following:

(a) $m^{*,J}(\overline{E}) = m^{*,J}(E)$, where $\overline{E}$ denotes the closure of E.
(b) $m_{*,J}(\mathring{E}) = m_{*,J}(E)$, where $\mathring{E}$ denotes the interior of E.
(c) E is Jordan measurable if and only if $m^{*,J}(\partial E) = 0$, where ∂E denotes the boundary of E.

Problem 6. Let $E := \mathbb{Q} \cap [0, 1]$. Prove the following

(a) $m_{*,J}(E) = 0$.
(b) $m^{*,J}(E) = 1$.

Conclude that E is not Jordan measurable.

More problems to be added after the Tuesday lecture.