THIRD WEEK HOMEWORK ASSIGNMENT

Problem 1. Prove that the elementary measure is monotone and subadditive. In other words, show that the map $m\colon \mathcal{E}(\mathbb{R}^d) \to \mathbb{R}$ satisfies the following:

(i) For all elementary sets E and F, if $E \subset F$ then $m(E) \leq m(F)$.
(ii) For all elementary sets E and F, $m(E \cup F) \leq m(E) + m(F)$.

Explain why the same properties hold for the Jordan measure.

Problem 2. Prove that the Cantor set is Jordan measurable and that its Jordan measure is 0.

Problem 3. Prove that if E and F are Jordan measurable sets, then $E \cap F$, $E \setminus F$ and $E \triangle F$ are Jordan measurable as well.

Hint: Use the characterization of Jordan measurability in terms of elementary sets that approximate from within and from without. Then everything reduces to some (Boolean) operations with sets.

Perhaps the trickiest part is showing that $E \setminus F$ is Jordan measurable. You may use (after verifying that they are indeed true) the following relations between sets.

If $A \subset E \subset B$ and $C \subset F \subset D$, then

$$A \setminus D \subset E \setminus F \subset B \setminus C.$$

Moreover,

$$(B \setminus C) \setminus (A \setminus D) \subset (B \setminus A) \cup (D \setminus C).$$

Problem 4. Prove that a bounded set $E \subset \mathbb{R}^d$ is Jordan measurable if and only if for all $\epsilon > 0$ there is an elementary set A such that $m^{*,J}(E \triangle A) < \epsilon$.

Problem 5. Let $E \subset \mathbb{R}^d$ be any bounded set. Prove the following:

(a) $m^{*,J}(\overline{E}) = m^{*,J}(E)$, where $\overline{E}$ denotes the closure of E.
(b) $m_{*,J}(\mathring{E}) = m_{*,J}(E)$, where $\mathring{E}$ denotes the interior of E.
(c) E is Jordan measurable if and only if $m^{*,J}(\partial E) = 0$, where ∂E denotes the boundary of E.

Hint: During the exercise section, someone made me aware of the fact that part (c) is quite tricky. The difficult part is to show that if $m^{*,J}(\partial E) = 0$ then E is Jordan measurable. Here is a hint.

Since $m^{*,J}(\partial E) = 0$, for every $\epsilon > 0$ there is an elementary set D with $\partial E \subset D$ and $m(D) < \epsilon$. We may assume that D is an open set (why?). Then $\overline{E} \setminus D$ is *compact* (why?).

Note that $\overline{E} \setminus D \subset \mathring{E}$, and by compactness we can find an elementary set B such that

$$\overline{E} \setminus D \subset B \subset \mathring{E}.$$

This implies $\overline{E} \subset B \cup D$. Since $B \cup D$ is an elementary set, we may then derive that $m^{*,J}(\overline{E}) \leq m_{*,J}(\mathring{E}) + \epsilon$. Let $\epsilon \to 0$, then use parts (a) and (b) to conclude that E is Jordan measurable.

This is of course a rough sketch of the proof, you have many details to fill in. Try it till next week on your own, and then I will post the detailed solution.

Problem 6. Let $E := \mathbb{Q} \cap [0, 1]$. Prove the following

(a) $m_{*,J}(E) = 0$.
(b) $m^{*,J}(E) = 1$.

Conclude that E is not Jordan measurable.

Problem 7. Prove that if f and g are Darboux integrable functions on $[a, b]$, then $f + g$ is Darboux integrable as well and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Hint: Use the characterization of the Darboux integrability in terms of "good" step functions (that approximate from below and above).

Problem 8. We have shown in class that if E is a Jordan measurable subset of $\mathbb{R}$, then its indicator function $\mathbf{1}_E(x)$ is Darboux integrable.

Prove that the reverse is also true. That is, prove that if the indicator function $\mathbf{1}_E(x)$ of some set E is Darboux integrable, then the set E must be Jordan measurable.