SECOND WEEK HOMEWORK ASSIGNMENT

Problem 1. Prove that the following sets are countable:

(a) The set of strictly positive integers $\mathbb{N}^\star = \{1, 2, \dots n, \dots\}$.
(b) The set of all even natural numbers.
(c) The set of all odd natural numbers.
(d) The set $\mathbb{Z}$ of all integers.

Problem 2. Prove that all (proper) intervals have the same cardinality, which we call the *continuum* cardinality.

Problem 3. [*] Prove Cantor-Schröder-Bernstein's theorem:

If $\operatorname{card} A \le \operatorname{card} B$ and $\operatorname{card} B \le \operatorname{card} A$ then $\operatorname{card} A = \operatorname{card} B$.

In other words, if there are a one-to-one function $f\colon A \to B$ and a one-to-one function $g\colon B \to A$, then there is a *bijection* $h\colon A \to B$.

Follow the steps in exercise 1.36 in the textbook.

Problem 4. Prove the following:

(a) Any *finite* subset of $\mathbb{R}$ is a closed set.
(b) $\mathbb{N}$ and $\mathbb{Z}$ are closed sets.
(c) $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are not closed sets.
(d) None of the sets above is open (except for the empty set).
(e) If $P(x)$ is a polynomial, then $\{x \in \mathbb{R} \colon P(x) > 0\}$ is open.

Problem 5. [*] Prove that any open set can be written as a disjoint union of at most countably many open intervals.

Problem 6. Prove that if F is a closed set, then its complement $F^{\complement} = \mathbb{R} \setminus F$ is an open set.

Problem 7. Prove (*without* using the Heine-Borel theorem) that if K is a compact set and if F is a closed set such that $F \subset K$, then F is a compact set as well.

Problem 8. Prove that any decreasing sequence of *closed* intervals whose lengths converge to 0 has a nonempty intersection.

In other words: let $I_1,\ I_2, \dots, I_n,\ I_{n+1}, \dots$ be closed intervals so that

$$I_1 \supset I_2 \supset \dots \supset I_n \supset I_{n+1} \supset \dots$$

and $\left|I_n\right| \to 0$ as $n \to \infty$.

Prove that

$$\bigcap_{n \ge 1} I_n \ne \emptyset\,.$$

This problem may be divided into the following easier steps.

(a) Pick in each interval I_n a point x_n and show that the sequence $\{x_n\}$ thus obtained is Cauchy, hence convergent.
(b) Let x be the limit of this sequence. Prove that x belongs to all intervals I_n (use the fact that closed intervals are closed sets).
(c) Prove that in fact $\bigcap_{n\geq 1} I_n$ only contains the point x.

Problem 9. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $c \in \mathbb{R}$, then

(a) The set $\{x \in \mathbb{R}\colon f(x) > c\}$ is open.
(b) The set $\{x \in \mathbb{R}\colon f(x) < c\}$ is open.
(c) The set $\{x \in \mathbb{R}\colon f(x) \in (c, d)\}$ is open.
(d) The set $\{x \in \mathbb{R}\colon f(x) = c\}$ is closed.
(e) The set $\{x \in \mathbb{R}\colon f(x) \leq c\}$ is closed.