THIRTEENTH WEEK HOMEWORK ASSIGNMENT

Let $(X, \mathcal{B}, \mu)$ be a measure space.

Problem 1. Let $f_1, f_2, \ldots : X \to \mathbb{R}$ be a sequence of measurable functions and assume that $f_n \to f$ in the L^1 norm as $n \to \infty$.

Prove that $f_n \to f$ in measure.

Hint: Use Markov's inequality.

Problem 2. Let $f_1, f_2, \ldots : X \to \mathbb{R}$ be a sequence of measurable functions and assume that $f_n \to f$ almost uniformly as $n \to \infty$.

Prove that $f_n \to f$ in measure.

Problem 3. Assume that $\mu(X) < \infty$.

Let $f_1, f_2, \ldots : X \to \mathbb{R}$ be a sequence of measurable functions and assume that $f_n \to f$ in the L^∞ norm as $n \to \infty$.

Prove that $f_n \to f$ in the L^1 norm.

Problem 4. Assume that $\mu(X) < \infty$.

Let $f_1, f_2, \ldots : X \to \mathbb{R}$ be a sequence of measurable functions and assume that $f_n \to f$ pointwise a.e. as $n \to \infty$.

Prove that $f_n \to f$ in measure.

Hint: You may assume (why?) that the pointwise convergence is everywhere, instead of a.e. Define the sets

$$A_N := \{x \in X : \left|f_n(x) - f(x)\right| \le \epsilon \quad \text{for all } n \ge N\}.$$

Verify that $A_N \nearrow X$, so ${A_N}^{\complement} \searrow \emptyset$.

Apply the downwards monotone convergence theorem for sets ...

Problem 5. Prove the differentiation theorem for continuous functions with compact support. More precisely, prove the following: if $f : [a, b] \to \mathbb{R}$ is a continuous function, then for every $x \in (a, b)$ we have

$$\lim_{h \to 0^+} \frac{1}{h} \int_{[x, x+h]} f(t)\, dt = f(x).$$

Hint: You could derive this using the fundamental theorem of calculus I. You could also derive it more directly (and I would prefer that) simply by using the fact that a continuous function on a compact set is uniformly continuous.

Problem 6. (a) Compute the limit

$$\lim_{n\to\infty} \int_0^1 e^{-x^n}\, dx.$$

(b) Show that the function $g\colon [1,\infty) \to \mathbb{R}$, $g(x) := e^{-x}$ is absolutely integrable. Then compute the limit

$$\lim_{n\to\infty} \int_1^\infty e^{-x^n}\, dx.$$

Hint: In part (a) use the monotone convergence theorem. In part (b) use the dominated convergence theorem.

Problem 7. Show that the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) := e^{-x^2}$ is absolutely integrable (relative to the Lebesgue measure).

Problem 8. Define the sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := e^{-x^2} [\sin(nx)]^n.$$

Show the following:

(a) $f_n \to 0$ pointwise a.e. (relative to the Lebesgue measure).

(b) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) dx = 0.$

(c) Prove also that $f_n \to 0$ in the L^1 norm (so in measure too).

Hint: It is helpful to remember that $\left|\sin y\right| \leq 1$ for all y, and also that $\sin y = \pm 1$ does not occur often.

For part (b) use the dominated convergence theorem and the previous problem. In part (c) you have almost nothing left to do, if you realize that parts (a) and (b) are still true with $\left|f_n(x)\right|$ instead of $f_n(x)$.