TWELFTH WEEK HOMEWORK ASSIGNMENT

Let $(X, \mathcal{B}, \mu)$ be a measure space.

Problem 1. Prove that if $f, g \colon X \to \mathbb{R}$ are two measurable functions, and if $f \le g$ μ-a.e., then

$$\int_X f \, d\mu \le \int_X g \, d\mu.$$

Note: Do this in full detail, by starting with simple functions, then considering non-negative functions, then using the splitting into positive and negative parts.

Problem 2. Prove Markov's inequality in an abstract measure space: if $f \colon X \to \mathbb{R}$ is measurable, then for every $\lambda > 0$ we have

$$\mu\left(\{x \in X \colon \left|f(x)\right| \ge \lambda\}\right) \le \frac{1}{\lambda} \int_X \left|f\right| d\mu\,.$$

Problem 3. Let $f \in L^1(X, \mu)$. Use Markov's inequality to show that all level sets $\{x \in X \colon \left|f(x)\right| \ge \lambda\}$ have finite measure. Then conclude that $\left|f\right| < \infty$ μ-a.e.

Problem 4. Prove that if $(X, \mathcal{B}, \mu)$ is a *finite* measure space, meaning that $\mu(X) < \infty$, and if $f_n \colon X \to [0, \infty]$, $n \ge 1$ is a sequence of measurable functions such that $f_n \to f$ *uniformly*, then

$$\int_X f_n \, d\mu \to \int_X f \, d\mu.$$

Problem 5. Let $0 \le f_1 \le f_2 \le \dots$ be an increasing sequence of measurable, non-negative functions on X. Let E be any $\mathcal{B}$-measurable set. Prove that

$$\lim_{n \to \infty} \int_E f_n \, d\mu = \int_E \left(\lim_{n \to \infty} f_n\right) d\mu\,.$$

Note: This is very similar to the statement of the monotone convergence theorem (MCT). The only difference is that the integration happens on a subset E and not on the whole space X. You need to find a way to apply the MCT.

Problem 6. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\phi \colon X \to [0,1]$ be a measurable function. For each $n \geq 1$ define the functions $f_n := \phi^{1/n}$ and $g_n := \phi^n$.

(a) Prove that the functions f_n and g_n are all measurable.
(b) Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \mu\left(\{x \in X \colon f(x) \neq 0\}\right).$$

(c) Prove that

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \mu\left(\{x \in X \colon f(x) = 1\}\right).$$

Hint: In part (a) use the fact that power functions are continuous.

In parts (b) and (c) use the MCT. Keep in mind that $0 \leq f(x) \leq 1$, which should tell you that the sequences f_n and g_n are monotonic.

Problem 7. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\phi \colon X \to [0, \infty]$ be a measurable function. Define the map $\nu \colon \mathcal{B} \to [0, \infty]$ as

$$\nu(A) := \int_A \phi \, d\mu = \int_X \phi \cdot \mathbf{1}_A \, d\mu \quad \text{for every } A \in \mathcal{B}.$$

(a) Prove that ν is a measure on $(X, \mathcal{B})$.
(b) Prove that if $f \colon X \to [0, \infty]$ is a measurable function, then

$$\int_X f \, d\nu = \int_X f \cdot \phi \, d\mu.$$

Hint: For part (a), just to warm up, begin by proving *finite* additivity of the map ν. To prove countable additivity, use Tonelli's theorem (the first consequence of the MCT).

Part (b) will be done in steps, starting from the simplest type of function f, to the most general.

i. Assume that f is an indicator function: $f = \mathbf{1}_E$ for some $E \in \mathcal{B}$.

ii. Assume that f is a simple function: $f = c_1\mathbf{1}_{E_1} + \ldots + c_k\mathbf{1}_{E_k}$.

iii. For the general case, f measurable, use the fact that there is an increasing sequence $f_n \nearrow f$ of non-negative simple functions. Apply step **ii.** to f_n and then the MCT.