

TWELFTH WEEK HOMEWORK ASSIGNMENT

Let (X, \mathcal{B}, μ) be a measure space.

Problem 1. Prove that if $f, g: X \rightarrow \mathbb{R}$ are two measurable functions, and if $f \leq g$ μ -a.e., then

$$\int_X f d\mu \leq \int_X g d\mu.$$

Note: Do this in full detail, by starting with simple functions, then considering non-negative functions, then using the splitting into positive and negative parts.

Problem 2. Prove Markov's inequality in an abstract measure space: if $f: X \rightarrow \mathbb{R}$ is measurable, then for every $\lambda > 0$ we have

$$\mu(\{x \in X: |f(x)| \geq \lambda\}) \leq \frac{1}{\lambda} \int_X |f| d\mu.$$

Problem 3. Let $f \in L^1(X, \mu)$. Use Markov's inequality to show that all level sets $\{x \in X: |f(x)| \geq \lambda\}$ have finite measure. Then conclude that $|f| < \infty$ μ -a.e.

Problem 4. Prove that if (X, \mathcal{B}, μ) is a *finite* measure space, meaning that $\mu(X) < \infty$, and if $f_n: X \rightarrow [0, \infty]$, $n \geq 1$ is a sequence of measurable functions such that $f_n \rightarrow f$ *uniformly*, then

$$\int_X f_n d\mu \rightarrow \int_X f d\mu.$$

Problem 5. Let $0 \leq f_1 \leq f_2 \leq \dots$ be an increasing sequence of measurable, non-negative functions on X . Let E be any \mathcal{B} -measurable set. Prove that

$$\lim_{n \rightarrow \infty} \int_E f_n d\mu = \int_E \left(\lim_{n \rightarrow \infty} f_n \right) d\mu.$$

Note: This is very similar to the statement of the monotone convergence theorem (MCT). The only difference is that the integration happens on a subset E and not on the whole space X . You need to find a way to apply the MCT.

Problem 6. Let (X, \mathcal{B}, μ) be a measure space and let $\phi: X \rightarrow [0, 1]$ be a measurable function. For each $n \geq 1$ define the functions $f_n := \phi^{1/n}$ and $g_n := \phi^n$.

- (a) Prove that the functions f_n and g_n are all measurable.
 (b) Prove that

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \mu(\{x \in X : \phi(x) \neq 0\}).$$

- (c) Prove that

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \mu(\{x \in X : \phi(x) = 1\}).$$

Hint: In part (a) use the fact that power functions are continuous.

In parts (b) and (c) use the MCT. Keep in mind that $0 \leq f(x) \leq 1$, which should tell you that the sequences f_n and g_n are monotonic.

Problem 7. Let (X, \mathcal{B}, μ) be a measure space and let $\phi: X \rightarrow [0, \infty]$ be a measurable function. Define the map $\nu: \mathcal{B} \rightarrow [0, \infty]$ as

$$\nu(A) := \int_A \phi d\mu = \int_X \phi \cdot \mathbf{1}_A d\mu \quad \text{for every } A \in \mathcal{B}.$$

- (a) Prove that ν is a measure on (X, \mathcal{B}) .
 (b) Prove that if $f: X \rightarrow [0, \infty]$ is a measurable function, then

$$\int_X f d\nu = \int_X f \cdot \phi d\mu.$$

Hint: For part (a), just to warm up, begin by proving *finite* additivity of the map ν . To prove countable additivity, use Tonelli's theorem (the first consequence of the MCT).

Part (b) will be done in steps, starting from the simplest type of function f , to the most general.

- i. Assume that f is an indicator function: $f = \mathbf{1}_E$ for some $E \in \mathcal{B}$.
- ii. Assume that f is a simple function: $f = c_1 \mathbf{1}_{E_1} + \dots + c_k \mathbf{1}_{E_k}$.
- iii. For the general case, f measurable, use the fact that there is an increasing sequence $f_n \nearrow f$ of non-negative simple functions. Apply step ii. to f_n and then the MCT.