ELEVENTH WEEK HOMEWORK ASSIGNMENT

Problem 1. Prove that the sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \frac{x}{n}$$

converges to the function $f(x) = 0$ locally uniformly but not uniformly.

Problem 2. Prove that the sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} \frac{1}{n\,x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

converges to the function $f(x) = 0$ pointwise but not locally uniformly.

Problem 3. Verify that an atomic algebra is a σ-algebra.

Here is a reminder of how an atomic algebra is defined. Let X be a set and consider a partition

$$X = \bigcup_{\alpha \in I} A_\alpha$$

into disjoint subsets (we refer to these subsets as "atoms"). Let

$$\mathcal{A} := \Big\{ \bigcup_{\alpha \in J} A_\alpha \colon J \subset I \Big\}.$$

In other words, $\mathcal{A}$ consists of all possible unions of atoms.

Then $\mathcal{A}$ is a σ-algebra.

Problem 4. Consider an atomic algebra with a *countable* set of atoms. Explain how one can define a measure on this measurable space and verify that what you have defined is indeed a measure.

Hint: Begin by assigning to each atom a certain "mass".

Problem 5. Prove that the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ may be generated by either of the following collections of subsets of $\mathbb{R}^d$:

(a) All closed sets;
(b) All compact sets;
(c) All open balls;
(d) All boxes.

Problem 6. Let $(X, \mathcal{B}, \mu)$ be a measure space. Prove the monotone convergence theorem for sets:

(a) Upwards: if $E_n \nearrow E$ as $n \to \infty$, then $\mu(E_n) \to \mu(E)$ as $n \to \infty$.
(b) Downwards: if $E_n \searrow E$ as $n \to \infty$ and if $\mu(E_N) < \infty$ for some N then $\mu(E_n) \to \mu(E)$ as $n \to \infty$.

Problem 7. Let $(X, \mathcal{B})$ be a measurable space and let $f \colon X \to [0, \infty]$ be a function. Show that f is measurable if and only if all the level sets

$$\{x \in X \colon f(x) > \lambda\}$$

are $\mathcal{B}$-measurable.

Recall that we have defined f to be *measurable* if for every open set $U \subset [0, \infty]$, its pre-image $f^{-1}(U)$ is $\mathcal{B}$-measurable.

Problem 8. Let $(X, \mathcal{B})$ be a measurable space and let $f \colon X \to \mathbb{R}$ be a measurable function. Prove that for every interval $I \subset \mathbb{R}$, the pre-image $f^{-1}(I)$ is $\mathcal{B}$-measurable.

Note: This is of course true for *open* intervals, since they are open sets. The point is then to show that it must also be true for closed or semi-closed intervals.