TENTH WEEK HOMEWORK ASSIGNMENT

Problem 1. Prove that if $\{y_n\}_{n\geq 1} \subset \mathbb{R}$ is a sequence of real numbers and if $y_n \to y$ as $n \to \infty$, then

$$y_n^+ \to y^+ \quad \text{and} \quad y_n^- \to y^- \quad \text{as } n \to \infty.$$

Hint: Consider three separate cases: $y > 0$, $y < 0$ and $y = 0$.

Problem 2. Prove that $L^1(\mathbb{R}^d)$ is a normed space with the norm

$$\|f\|_{L^1} := \int_{\mathbb{R}^d} \big|f(x)\big|\, dx.$$

In other words:

(a) Show that the set of absolutely integrable functions is a subspace of the vector space of all functions on $\mathbb{R}^d$.
(b) Let $L^1(\mathbb{R}^d)$ be the space of absolutely integrable functions where two functions that are equal a.e. are identified (i.e. they are regarded as representing the same vector). Prove that $\|\cdot\|_{L^1}$ defined above satisfies the three axioms of a norm.

Problem 3. Prove that the Lebesgue integral is a linear map from $L^1(\mathbb{R}^d)$ to $\mathbb{R}$. In other words, show that

$$\int_{\mathbb{R}^d} (f+g) = \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g \quad \text{and} \int_{\mathbb{R}^d} cf = c\int_{\mathbb{R}^d} f,$$

for any absolutely integrable functions f and g and any constant $c \in \mathbb{R}$.

Hint: Use the same strategy employed to establish the linearity of the integral for simple *signed* functions, once the linearity was known for *unsigned* functions.

Problem 4. Prove the monotonicity of the integral. If $f, g \colon \mathbb{R}^d \to \mathbb{R}$ are absolutely integrable and if $f \leq g$ a.e. then

$$\int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} g.$$

Problem 5. Prove the following triangle inequality: if $f \colon \mathbb{R}^d \to \mathbb{R}$ is absolutely integrable then

$$\Big|\int_{\mathbb{R}^d} f\Big| \leq \int_{\mathbb{R}^d} |f|.$$

Problem 6. Prove the divisibility property: if $f\colon \mathbb{R}^d \to \mathbb{R}$ is absolutely integrable and if $E \subset \mathbb{R}^d$ is measurable, then

$$\int_{\mathbb{R}^d} f = \int_E f + \int_{E^{\complement}} f\,,$$

where $E^{\complement} = \mathbb{R}^d \setminus E$ is the complement of E and if S is any measurable set, $\int_S f := \int_{\mathbb{R}^d} f \cdot \mathbf{1}_S$.

Problem 7. Prove that if $\{f_n\}_{n\geq 1}$ is a sequence of measurable functions on $\mathbb{R}^d$ and if for a.e. $x \in \mathbb{R}^d$ we have

$$\lim_{n\to\infty} f_n(x) = f(x),$$

then the limit function $f(x)$ is measurable as well.

Hint: Use the same approach we described in class when proving the equivalent notions of measurability (namely the idea of the proof of Lemma 1.3.9 (iii) implies (v) in Tao's book).