Introduction

I assume you are familiar with finite sums

$$\sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$$

which can be defined by induction on n:

$$\sum_{k=1}^{0} a_k = 0, \qquad \sum_{k=1}^{n+1} a_k = \left(\sum_{k=1}^{n} a_k\right) + a_{n+1}, \qquad n = 0, 1, 2, \ldots$$

and the corresponding infinite sums

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

provided the limit exists.

We will find it useful to extend these notions to index sets other than $\{1, \ldots, n\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. These index sets will have no specified order; that will hardly make a difference for finite index sets, but for infinite index sets, the difference is profound. It turns out that only the theory of *absolutely convergent* sequences generalizes to arbitrary index sets.

Sums over sets

Defining

$$\sum_{\iota\in I} a_\iota$$

for some family $(a_\iota)_{\iota\in I}$ of real numbers with a *finite* index set I is unproblematic. Just list the members of I in some order as $I = \{\iota_1, \ldots, \iota_n\}$ and define

$$\sum_{\iota\in I} a_\iota = a_{\iota_1} + \cdots + a_{\iota_n}.$$

We can be more pedantic about it if you wish: Define

$$\sum_{\iota\in\emptyset} a_\iota = 0,$$

and if I is nonempty, pick any member $\iota' \in I$ and define

$$\sum_{\iota\in I} a_\iota = \left(\sum_{\iota\in I\setminus\{\iota'\}} a_\iota\right) + a_{\iota'},$$

proceeding by induction on the number of elements of I. You do, however, have to prove that this is *well defined* in that the answer does not depend on your choice of ι'. This will need to be done as part of the induction proof. For these notes, however, we will take the result for granted.

Sums of positive functions on infinite sets

If I is an *infinite* set, we shall at first define the sum only in the case that $a_\iota \geq 0$ for all $\iota \in I$. If so, define

$$\sum_{\iota\in I} a_\iota = \sup\Big\{\sum_{\iota\in F} a_\iota : F \subseteq I \text{ is finite}\Big\}.$$

If I is finite, the above equality certainly holds, since $\sum_{\iota\in F} a_\iota \leq \sum_{\iota\in I} a_\iota$ when $F \subseteq I$ and $a_\iota \geq 0$.

When the index set is the set of natural numbers, this turns out to be nothing new:

1 Proposition *Given a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative numbers,*

$$\sum_{n\in\mathbb{N}} a_n = \sum_{k=1}^{\infty} a_k.$$

Proof: For every $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} a_k = \sum_{k\in\{1,\ldots,n\}} a_k \leq \sum_{k\in\mathbb{N}} a_k$$

by the definition of the latter sum. Letting $n \to \infty$ we get

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k\in\mathbb{N}} a_k.$$

On the other hand, if $F \subset \mathbb{N}$ is finite then $F \subseteq \{1, \ldots, n\}$ for any $n \geq \max F$, and so

$$\sum_{k\in F} a_k \leq \sum_{k\in\{1,\ldots,n\}} a_k = \sum_{k=1}^{n} a_k.$$

Letting $n \to \infty$ in this inequality we get

$$\sum_{k\in F} a_k \leq \sum_{k=1}^{\infty} a_k,$$

and since this holds for all finite $F \subset \mathbb{N}$,

$$\sum_{k\in\mathbb{N}} a_k \leq \sum_{k=1}^{\infty} a_k,$$

and the proof is complete. ∎

Since our new sum is defined without regard to any ordering of the index set, it follows that the sum of a nonnegative sequence is independent of ordering:

2 Corollary *The sum of a sequence of nonnegative numbers is invariant under permutations of* $\mathbb{N}$.

In other words, let $\sigma\colon \mathbb{N}\to\mathbb{N}$ *be a bijective map. Then*

$$\sum_{k=1}^{\infty} a_{\sigma(k)} = \sum_{k=1}^{\infty} a_k.$$

Proof: In the following calculation, the supremum is over finite subsets of $\mathbb{N}$.

$$\sum_{j=1}^{\infty} a_{\sigma(j)} = \sum_{j\in\mathbb{N}} a_{\sigma(j)} = \sup_F \sum_{j\in F} a_{\sigma(j)} = \sup_F \sum_{k\in\sigma(F)} a_k = \sum_{k\in\mathbb{N}} a_k = \sum_{k=1}^{\infty} a_k,$$

where the next to last equality holds because the sets $\sigma(F)$ where F is a finite subset of $\mathbb{N}$ are precisely the finite subsets of $\mathbb{N}$. ∎

Countability in the index set

3 Proposition *If* $\sum_{\iota\in I} a_\iota < \infty$ *where* $(a_\iota)_{\iota\in I}$ *is an indexed family with* $a_\iota \ge 0$ *for all* $\iota\in I$*, then* $\{\iota\in I\colon a_\iota > 0\}$ *is countable.*

Proof: Write $S = \sum_{\iota\in I} a_\iota$, and let $F_\varepsilon = \{\iota\in I\colon a_\iota \ge \varepsilon\}$, when $\varepsilon > 0$. Since $\sum_{\iota\in F_\varepsilon} a_\iota \le S$, it is clear that F_ε has at most S/ε elements. So F_ε is finite. But then

$$\{\iota\in I\colon a_\iota > 0\} = \bigcup_{n\in\mathbb{N}} F_{1/n}$$

which is a countable union of finite sets, so this set is countable. ∎

Absolutely convergent sums

Now consider an indexed family $(a_\iota)_{\iota\in I}$ of real numbers without restriction on the signs of a_ι.

For any real number a, we let

$$a^+ = \begin{cases} a & a\ge 0,\\ 0 & a<0,\end{cases} \qquad a^- = \begin{cases} 0 & a\ge 0,\\ -a & a<0,\end{cases}$$

so that $a^\pm \ge 0$, $a^+a^- = 0$, $a = a^+ - a^-$, and $|a| = a^+ + a^-$.

We call the sum $\sum_{\iota\in I} a_\iota$ *absolutely convergent* if $\sum_{\iota\in I} |a_\iota| < \infty$, or equivalently, if both sums $\sum_{\iota\in I} a_\iota^\pm$ are finite. In this case, we can define

$$\sum_{\iota\in I} a_\iota = \sum_{\iota\in I} a_\iota^+ - \sum_{\iota\in I} a_\iota^-.$$

This definition of the sum even makes sense if just *one* of the sums $\sum_{\iota\in I} a_\iota^\pm$ is finite. In that case the sum will be $\pm\infty$, the sign depending on which of the right hand sums is infinite.

Double sums

4 Proposition *Assume* $a_{ij} \ge 0$ *whenever* $i\in I$ *and* $j\in J$*. Then*

$$\sum_{i\in I}\sum_{j\in J} a_{ij} = \sum_{(i,j)\in I\times J} a_{ij} = \sum_{j\in J}\sum_{i\in I} a_{ij}.$$

Proof: We only need to show the first equality, as the second then follows by interchanging I and J.

First, let $F\subseteq I$ be finite. For each $i\in F$, let $G_i\subseteq J$ be finite, and let $H = \bigcup_{i\in F}(\{i\}\times G_i)$. Then H is a finite subset of $I\times J$, so

$$\sum_{i\in F}\sum_{j\in G_i} a_{ij} = \sum_{(i,j)\in H} a_{ij} \le \sum_{(i,j)\in I\times J} a_{ij}.$$

In this inequality, take the supremum over all choices of G_i to get

$$\sum_{i\in F}\sum_{j\in J} a_{ij} \le \sum_{(i,j)\in I\times J} a_{ij}.$$

This is really a special case of the equality

$$\sup(A_1+\cdots+A_n) = \sup A_1 + \cdots + \sup A_n$$

where $A_1,\ldots,A_n\subseteq\mathbb{R}$.

Next, take the supremum over all finite $F\subseteq I$ to conclude

$$\sum_{i\in I}\sum_{j\in J} a_{ij} \le \sum_{(i,j)\in I\times J} a_{ij}.$$

To obtain the opposite inequality, notice that if $H\subseteq I\times J$ is finite, then there are finite subsets $F\subseteq I$ and $G\subseteq J$ with $H\subseteq I\times J$, and so

$$\sum_{(i,j)\in H} a_{ij} \le \sum_{(i,j)\in F\times G} a_{ij} = \sum_{i\in F}\sum_{j\in G} a_{ij} \le \sum_{i\in F}\sum_{j\in J} a_{ij} \le \sum_{i\in I}\sum_{j\in J} a_{ij},$$

and the proof is completed by taking the supremum over all H. ∎

Epilogue

When we get around to defining the Lebesgue integral, our approach will be very similar to the present one.

We will define the integral first for *simple* functions, which only take a finite number of values. Then we will define the integral of a nonnegative function f as

$$\int f\,d\lambda = \sup_{0\le\varphi\le f} \int \varphi\,d\lambda$$

where the supremum is taken over simple functions φ. We then extend the integral to functions f with $\int |f|\,d\lambda < \infty$ by taking positive and negative parts, integrating, and subtracting: $\int f\,d\lambda = \int f^+\,d\lambda - \int f^-\,d\lambda$.

To spell out the analogy more carefully, we could have defined

$$\sum_{\iota\in I} a_\iota = \sup_{0\le\varphi\le a} \sum_{\iota\in I} \varphi_\iota$$

where the supremum is taken over indexed families $(\varphi_\iota)_{\iota\in I}$ where $0 \le \varphi_\iota \le a_\iota$ for all ι, and $\varphi_\iota > 0$ only for a finite number of indices ι.