Semialgebras

A *semialgebra* on a set Ω is a set $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ so that

- $\emptyset \in \mathcal{C}$,
- $A \cap B \in \mathcal{C}$ if $A, B \in \mathcal{C}$,
- If $\emptyset \neq A \in \mathcal{C}$ then $A^c = \bigsqcup_{k=1}^{n} B_k$ for a finite sequence $B_1, \ldots, B_n$ in $\mathcal{C}$.

The textbook doesn't include the first axiom ($\emptyset \in \mathcal{C}$), and for its troubles doesn't get any more generality except for the possibility that $\mathcal{C} = \{\Omega\}$. I don't want to bother with that possibility.

Premeasures

A *premeasure* on a semialgebra $\mathcal{C}$ is a function $\iota\colon \mathcal{C} \to [0,\infty]$ so that

- $\iota(\varnothing) = 0$,
- $\iota\left(\bigsqcup_{k=1}^{m} A_k\right) = \sum_{k=1}^{m} \iota(A_k)$ if $A_1, \ldots, A_m$, and $\bigsqcup_{k=1}^{m} A_k$ all belong to sC,
- $\iota(A) \leq \sum_{n=1}^{\infty} \iota(C_n)$ if $A \in \mathcal{C}$, each $C_n \in \mathcal{C}$, and $A \subseteq \bigcup_{n=1}^{\infty} C_n$.

Recall that $\sqcup$ is used for a union of *pairwise disjoint* sets only.

Outer measure

If ι is a premeasure on a semialgebra $\mathcal{C}$, we can define an *outer measure* μ^* on Ω by

$$\mu^*(A) = \inf \sum_{n=1}^{\infty} \iota(C_n),$$

with the infimum taken over all sequences (C_n) in $\mathcal{C}$ with $A \subseteq \bigcup_{n=1}^{\infty} C_n$.

To see that it is an outer measure, note that $\mu^*(\emptyset) = 0$ is trivial, and so is monotonicity ($\mu^*(A) \le \mu^*(B)$ if $A \subseteq B$).

For countable subadditivity, let (A_k) be a sequence of subsets of Ω, assume $\mu^*(A_k) < \infty$, let $\varepsilon > 0$, and select $C_{kn} \in \mathcal{C}$ with $A_k \subseteq \bigcup_n C_{kn}$ and $\sum_n \iota(C_{kn}) < \mu^*(A_n) + 2^{-n}\varepsilon$. Then $\bigcup_k A_k \subseteq \bigcup_k \bigcup_n C_{kn}$, and so

$$\mu^*\Big(\bigcup_k A_k\Big) \le \sum_k \sum_n \iota(C_{kn}) < \sum_k \big(\mu^*(A_k) + 2^{-k}\varepsilon\big) \le \sum_k \mu^*(A_k) + \varepsilon,$$

and therefore

$$\mu^*\Big(\bigcup_k A_k\Big) \le \sum_k \mu^*(A_k)$$

since $\varepsilon > 0$ was arbitrary.

We also note that

$$\mu^*(A) = \iota(A) \qquad \text{for } A \in \mathcal{C}.$$

This follows from the third axiom of premeasures, which simply states that $\iota(A) \le \mu^*(A)$; the opposite inequality following from the definition of μ^* and $A = A \cup \emptyset \cup \emptyset \cup \cdots$.

Measurability and extension

We find that every $E \in \mathcal{C}$ is μ^*-measurable.

Proof. For write $E^c = F_1 \sqcup \cdots \sqcup F_m$ where $F_k \in \mathcal{C}$ are pairwise disjoint. Write also $F_0 = E$, so that $\Omega = F_0 \sqcup F_1 \sqcup \cdots \sqcup F_m$. I *claim* that

$$\mu^*(W) \geq \sum_{k=0}^{m} \mu^*(W \cap F_k)$$

for any $W \subseteq \Omega$. If this is true, then we get

$$\mu^*(W) \geq \mu^*(W \cap E) + \sum_{k=1}^{m} \mu^*(W \cap F_k) \geq \mu^*(W \cap E) + \mu^*(W \setminus E)$$

by subadditivity, since $W \setminus E = (W \cap F_1) \cup \cdots \cup (W \cap F_m)$.

To prove the claim, then, assume $C_n \in \mathcal{C}$ and $W \subseteq \bigcup_{n=1}^{\infty} C_n$, and note that

$$\begin{aligned}
\sum_{n=1}^{\infty} \iota(C_n) &= \sum_{n=1}^{\infty} \sum_{k=0}^{m} \iota(C_n \cap F_k) \qquad C_n = \bigsqcup_{k=0}^{m} (C_n \cap F_k) \\
&= \sum_{k=0}^{m} \sum_{n=1}^{\infty} \iota(C_n \cap F_k) \\
&\geq \sum_{k=0}^{m} \mu^*(W \cap F_k) \qquad W \cap F_k \subseteq \bigcup_{n=1}^{\infty} C_n \cap F_k
\end{aligned}$$

and since this applies to any such sequence (C_n), the desired conclusion follows.

Uniqueness of the extension

Assume that ι is *σ-finite* in the following sense: There exists a sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ with $\iota(C_n) < \infty$ and $\Omega = \bigcap_{n\in\mathbb{N}} C_n$.

If ι is σ-finite and μ, ν are measures on $\sigma(\mathcal{C})$ with $\mu|_{\mathcal{C}} = \nu_{\mathcal{C}} = \iota$, then $\mu = \nu$.

Proof. First, we prove a simple *Lemma*: *The algebra generated by $\mathcal{C}$ consists of the sets which can be written as finite disjoint unions of members of $\mathcal{C}$.*

To see this, it is enough to show that the described set is an algebra of sets. It is closed under *intersections*, since

$$\Big(\bigsqcup_{j=1}^{m} A_j\Big) \cap \Big(\bigsqcup_{k=1}^{n} B_k\Big) = \bigsqcup_{j,k} A_j \cap B_k,$$

the union on the right being over all $(j,k) \in \mathbb{N}^2$ with $1 \le j \le m$ and $1 \le k \le n$. It is also closed under *complements*, for

$$\Big(\bigsqcup_{j=1}^{m} A_j\Big)^c = \bigcap_{j=1}^{m} A_j^c,$$

and each set on the right is a finite disjoint uniont of members of $\mathcal{C}$, so the first part of the proof applies.

Now to the proof of the main result: It follows from the lemma that μ and ν agree on all members of the algebra $\mathcal{A}_0$ generated by $\mathcal{C}$.

We will show that the two measures agree on the monotone class generated by $\mathcal{A}_0$. By the monotone class theorem, this is the same as the σ-algebra generated by $\mathcal{A}_0$, so that will finish the proof.

With $\Omega = \bigcap_{n\in\mathbb{N}} C_n$ with $C_n \in \mathcal{C}$ and $\iota(C_n) < \infty$, we can write $D_n = C_1 \cup \ldots \cup C_n$, so $D_n \in \mathcal{A}_0$, $D_n \nearrow \Omega$ (by which we mean $D_n \subseteq D_{n+1}$ for all n, and $\Omega = \bigcup_{n\in\mathbb{N}} D_n$), and $\mu(D_n) = \nu(D_n) < \infty$ for all $n \in \mathbb{N}$.

Let

$$\mathcal{D} = \big\{E \in \sigma(\mathcal{C}) : \mu(E \cap D_n) = \nu(E \cap D_n) \text{ for all } n \in \mathbb{N}\big\}.$$

Then $\mathcal{A}_0 \subseteq \mathcal{D} \subseteq \sigma(\mathcal{C})$, and $\mathcal{D}$ is a monotone class:

First assume $E_k \in \mathcal{D}$ and $E_k \nearrow E$. Then for each $n \in \mathbb{N}$, $E_k \cap D_n \nearrow E \cap D_n$, so

$$\mu(E \cap D_n) = \lim_{k\to\infty} \mu(E_k \cap D_n) = \lim_{k\to\infty} \nu(E_k \cap D_n) = \nu(E \cap D_n)$$

A similar argument applies of $E_k \searrow E$. In this case, we require the estimate $\mu(E_k \cap D_n) \le \mu(D_n) < \infty$ (and the same for ν) in order to draw the conclusion $\mu(E \cap D_n) = \nu(E \cap D_n)$. We have shown that $\mathcal{D}$ is a monotone class, and therefore, $\mathcal{D} = \sigma(\mathcal{C})$.

For any $E \in \sigma(\mathcal{C})$, we note that $E \cap D_n \nearrow E$, and therefore

$$\mu(E) = \lim_{n\to\infty} \mu(E \cap D_n) = \lim_{n\to\infty} \nu(E \cap D_n) = \nu(E),$$

so $\mu = \nu$, and the proof is complete.