Littlewood's principles

Littlewood's principles exist in many variants. In one variant they look like this, in no particular order:

- Any Lebesgue measurable set of finite measure is almost compact
- Any Lebesgue measurable function is almost continuous
- Pointwise convergence on a compact set is almost uniform

Common to all of these is that *almost* is taken in a measure theoretic sense, with exceptional set having arbitrarily small, but most likely nonzero, measure.

Principle: Measurable sets of finite measure are almost compact

1 Theorem *Let $E \subseteq \mathbb{R}$ with $\lambda^*(E) < \infty$. The E is Lebesgue measurable if and only if for each $\varepsilon > 0$ there is some compact set K and a set S with $E = K \sqcup S$ and $\lambda^*(S) < \varepsilon$.*

Proof: Assume first that $E \in \mathcal{M}$. Then since $[-n,n] \cap E \nearrow E$ when $n \to \infty$, we can pick n large enough so $\lambda(E \setminus [-n,n]) < \varepsilon$. Write $F = [-n,n] \cap E$ and $G = [-n,n] \setminus E$. Pick an open set $V \supseteq G$ with $\lambda(V \setminus G) < \varepsilon$, and let $K = [-n,n] \setminus V$. Then K is bounded and closed, so K is compact. Now

$$E = K \sqcup \underbrace{(V \cap F) \sqcup (E \setminus [-n,n])}_{\text{call this } S},$$

and then $V \cap F \subseteq V \setminus G$, so $\lambda(S) \le \lambda(V \cap F) + \lambda(E \setminus [-n,n]) < 2\varepsilon$.

Conversely, if for each n there is a compact K_n and a set S_n with $E = K \sqcup K_n \cup S_n$ and $\lambda^*(S_n) < 1/n$, let $F = \bigcup_{n\in\mathbb{N}} K_n$ and $S = \bigcap_{n\in\mathbb{N}} S_n$. Then $E = F \sqcup S$, $F \in \mathcal{M}$, and $\lambda^*(S) \le \lambda^*(S_n) < 1/n$ for any n, hence $\lambda^*(S) = 0$, so $S \in \mathcal{M}$, and so $E \in \mathcal{M}$. ∎

Most of the time, we will not mention the set S, but instead state the conclusion as follows as $\lambda(E \setminus K) < \varepsilon$, and think of K as being *almost all* of E, though this is a weaker *almost* than the one found in the standard phrase *almost every* (a.e.). In the other two Principles, we shall find such sets with additional desirable properties.

Often, we shall resort to the following trick: If we can find compact $K_n \subseteq E$ with $\lambda(E \setminus K) < \varepsilon_n$ for each n, where $\varepsilon_n > 0$, then $K = \bigcap_{n\in\mathbb{N}} K_n$ is compact, and $\lambda(E \setminus K) < \sum_{n\in\mathbb{N}} \varepsilon_n$. In particular, this trick is very useful with $\varepsilon_n = 2^{-n}\varepsilon$ for some fixed $\varepsilon > 0$.

It is tempting to state this as follows: *The intersection of a countable union of compacts, each of which is almost all of E, is almost all of E.* But then we should not forget that the differences will add up, it's just that we can arrange for a countable sum of small positive numbers to be small if we wish – and we *do* wish.

Principle: Measurable functions are almost continuous

This is also known as **Lusin's theorem**. It has many variants. We shall state and prove two of them here.

2 Theorem (Lusin) *Let $f\colon E \to \mathbb{C}$ be Lebesgue measurable, with $E \in \mathcal{M}$ and $\lambda(E) < \infty$. Then for any $\varepsilon > 0$ there is a compact set $K \subseteq E$ with $\lambda(E \setminus K) < \varepsilon$ so that $f|_K$ is continuous.*

Proof: We prove it first for simple functions, say $f = \sum_{k=1}^{n} a_k [A_k]$ with $A_k \in \mathcal{M}$ and $E = \bigsqcup_{k=1}^{n} A_k$. Pick compact $K_k \subseteq A_k$ with $\lambda(A_k \setminus K_k) < \varepsilon/n$, and set $K = \bigsqcup_{k=1}^{n} K_k$. Then f is constant on each K_k, and since disjoint compact sets have a mutual positive distance, it follows that $f|_K$ is continuous. Furthermore $E \setminus K = \bigsqcup_{k=1}^{n} (A_k \setminus K_k)$, so $\lambda(E \setminus K) < n \cdot (\varepsilon/n) = \varepsilon$.

Next, we consider a measurable function $f \ge 0$. We may allow the value $f(x) = \infty$, but must assume $f(x) < \infty$ a.e. Then, with of Theorem 1 applied to a set $f^{-1}([0,n])$ for large enough n, we may find a compact $L \subseteq E$ with $\lambda E \setminus L < \varepsilon$ so that $f|_L$ is bounded.

Now recall the proof of the monotone convergence theorem, where we showed that this sequence (ϕ_n) of simple functions converges to f pointwise:

$$\phi_n(x) = \sum_{k=1}^{n\cdot 2^n} 2^{-n} k \cdot \big[2^{-n} k \le f(x) < 2^{-n}(k+1)\big].$$

Moreover, the convergence will be *uniform* on L. In fact, when n is large enough (so $f \le n$ on L), we find $|f - \phi_n| \le 2^{-n}$ on L. Now we make good use of the remarks after Theorem 1: Let $K_n \subseteq E$ be compact with $\lambda(E \setminus K_n) < 2^{-n}\varepsilon$, and define $K = L \cap \big(\bigcap_{n\in\mathbb{N}} K_n\big)$. Then $\lambda(E \setminus L) < 2\varepsilon$. And since $\phi_n|_K$ is continuous for each n and the convergence is uniform on K, then the limit $f|_K$ is continuous.

Extending the result first to arbitrary measurable extended real-valued functions f with $|f| < \infty$ a.e., and then measurable complex-valued functions, is straightforward. ∎

3 Theorem (Lusin) *Let f be either an extended real-valued function on $\mathbb{R}$ with $|f| < \infty$ a.e. or a complex-valued function on $\mathbb{R}$. Then f is measurable if, and only if, for each $\varepsilon > 0$ there is a continuous function g on $\mathbb{R}$ and a closed set $F \subseteq \mathbb{R}$ with $\lambda(\mathbb{R} \setminus F) < \varepsilon$ so that $f = g$ on F.*

Proof: We start with the easy part, the "if" direction. So we assume that the given condition holds. Then for each $n \in \mathbb{N}$ we can find a continuous function g_n and a closed set closed set F_n with $\lambda(\mathbb{R} \setminus F_n) < 2^{-n}$ so that $f_n = g_n$ on F_n. Let $h_n = [F_n] f_n$. This function is measurable, since it equals $[F_n] g_n$, and both $[E_n]$ and g_n are measurable. Let $H = \bigcup_{n\in\mathbb{N}} \bigcap_{k\ge n} F_k$. Then $\mathbb{R} \setminus H = \bigcap_{n\in\mathbb{N}} \bigcup_{k\ge n} (\mathbb{R} \setminus F_k)$, and $\lambda\big(\bigcup_{k\ge n} (\mathbb{R} \setminus F_k)\big) < 2^{1-n}$, so $\lambda(\mathbb{R} \setminus H) = 0$. Also $h_n \to f$ pointwise on H, (in fact

$h_n(x) = f(x)$ for n large enough if $x \in H$), hence $f|_H$ is measurable, and it follows that f is measurable on $\mathbb{R}$.

For the converse, assume that f is measurable. Let $\varepsilon > 0$. For each n, pick a compact set $K_n \subseteq (-n, n)$ with $\lambda((-n, n) \setminus K) \leq 2^{-n}\varepsilon$ so that f_{K_n} is continuous. Let

$$F = \bigcap_{n\in\mathbb{N}} \big(K_n \cup (\mathbb{R} \setminus (-n, n))\big).$$

Then F is closed, since it is the intersection of closed sets. Also

$$\mathbb{R} \setminus F = \bigcup_{n\in\mathbb{N}} \big((-n, n) \setminus K_n\big),$$

so $\lambda(\mathbb{R} \setminus F) < \varepsilon$. Moreover for any n, $F \cap (-n, n) \subseteq K_n$, so $f|_F$ is continuous on $(-n, n)$. Since continuity is a local property, $f|_F$ is continuous.

It remains to extend the continuous function $f|_F$ to a continuous function g on $\mathbb{R}$. The open set $\mathbb{R} \setminus F$ is a disjoint union of intervals (the components of $\mathbb{R} \setminus F$), whose finite endpoints belong to F. Extend $f|_F$ by linear interpolation between the endpoints for any bounded component of $\mathbb{R} \setminus F$, and by setting it to a constant (equal to the value at one endpoint) on any unbounded component. The continuity of the function so defined is left as an exercise. ∎

Principle: Pointwise convergence is almost uniform

– but only on sets of *finite measure* – the obvious counterexample otherwise being $f_n(x) = x/n$ on $\mathbb{R}$: Clearly, $f_n \to 0$ pointwise, but the convergence is *far* from uniform.

4 Theorem (Egorov) *Assume a sequence of measurable functions on a Lebesgue measurable set $E \subset \mathbb{R}$ with $\lambda(E) < \infty$ is given. If the sequence converges pointwise with a finite limit a.e. on E, then for any $\varepsilon > 0$ there is a compact set K with $\lambda(E \setminus K) < \varepsilon$ on which the sequence converges uniformly.*

Proof: Since the limit is finite a.e., we can subtract it from the given sequence. In other words, we may assume that the given sequence (f_n) converges to 0 pointwise a.e. Define

$$g_n(x) = \sup_{k\ge n} f_k(x).$$

Then $g_n \searrow 0$ a.e., and we only need to find a compact set with small complement on which this convergence is uniform.

To this end, let K be a compact set with $\lambda(E \setminus K) < \varepsilon$ so that every g_n is continuous on K and $g_n \searrow 0$ on K.

The proof is now finished by an appeal to Dini's theorem, which we state and prove below. ∎

5 Theorem (Dini) *A monotone sequence of continuous functions converging pointwise on a compact set, converges uniformly.*

Proof: We may assume, without loss of generality, that the given sequence (g_n) decreases pointwise to 0 on a compact set K.

Assume the convergence is *not uniform*. Then there is some $\varepsilon > 0$ so that for every n, there is some $x_n \in K$ with $g_n(x_n) > \varepsilon$. (We only need this for infinitely many n, but the monotonicity of the sequence buys us this much extra.)

Let x be an accumulation point of the sequence (x_n). Since $g_n(x) \to 0$, there is some n with $g_n(x) < \varepsilon$. And since g_n is continuous, there is some $\delta > 0$ so that $g_n(y) < \varepsilon$ whenever $|x - y| < \delta$. But if k is large enough then $|x - x_k| < \delta$, and in particular if also $k \ge n$ we then get $g_k(x_k) \le g_n(x_k) < \varepsilon$, which contradicts the choice of x_k so $g_k(x_k) \ge \varepsilon$.

An *alternative proof* is to note that if the convergence is not uniform, there is some $\varepsilon > 0$ so that the set $F_n = \{x \in K\colon g_n(x) \ge \varepsilon\}$ is nonempty for all n. But then this is a decreasing sequence of nonempty closed subsets of K, and hence nonempty by the compactness of K. If $x \in \bigcap_{n\in\mathbb{N}} F_n$, then $g_n(x) \ge 0$ for all n, which contradicts the pointwise convergence. ∎