

## Littlewood's principles

Littlewood's principles exist in many variants. In one variant they look like this, in no particular order:

- Any Lebesgue measurable set of finite measure is almost compact
- Any Lebesgue measurable function is almost continuous
- Pointwise convergence on a compact set is almost uniform

Common to all of these is that *almost* is taken in a measure theoretic sense, with exceptional set having arbitrarily small, but most likely nonzero, measure.

### Principle: Measurable sets of finite measure are almost compact

**1 Theorem** *Let  $E \subseteq \mathbb{R}$  with  $\lambda^*(E) < \infty$ . The  $E$  is Lebesgue measurable if and only if for each  $\varepsilon > 0$  there is some compact set  $K$  and a set  $S$  with  $E = K \sqcup S$  and  $\lambda^*(S) < \varepsilon$ .*

**Proof:** Assume first that  $E \in \mathcal{M}$ . Then since  $[-n, n] \cap E \nearrow E$  when  $n \rightarrow \infty$ , we can pick  $n$  large enough so  $\lambda(E \setminus [-n, n]) < \varepsilon$ . Write  $F = [-n, n] \cap E$  and  $G = [-n, n] \setminus E$ . Pick an open set  $V \supseteq G$  with  $\lambda(V \setminus G) < \varepsilon$ , and let  $K = [-n, n] \setminus V$ . Then  $K$  is bounded and closed, so  $K$  is compact. Now

$$E = K \sqcup \underbrace{(V \cap F) \sqcup (E \setminus [-n, n])}_{\text{call this } S},$$

and then  $V \cap F \subseteq V \setminus G$ , so  $\lambda(S) \leq \lambda(V \cap F) + \lambda(E \setminus [-n, n]) < 2\varepsilon$ .

Conversely, if for each  $n$  there is a compact  $K_n$  and a set  $S_n$  with  $E = K_n \sqcup S_n$  and  $\lambda^*(S_n) < 1/n$ , let  $F = \bigcup_{n \in \mathbb{N}} K_n$  and  $S = \bigcap_{n \in \mathbb{N}} S_n$ . Then  $E = F \sqcup S$ ,  $F \in \mathcal{M}$ , and  $\lambda^*(S) \leq \lambda^*(S_n) < 1/n$  for any  $n$ , hence  $\lambda^*(S) = 0$ , so  $S \in \mathcal{M}$ , and so  $E \in \mathcal{M}$ . ■

Most of the time, we will not mention the set  $S$ , but instead state the conclusion as follows as  $\lambda(E \setminus K) < \varepsilon$ , and think of  $K$  as being *almost all* of  $E$ , though this is a weaker *almost* than the one found in the standard phrase *almost every* (a.e.). In the other two Principles, we shall find such sets with additional desirable properties.

Often, we shall resort to the following trick: If we can find compact  $K_n \subseteq E$  with  $\lambda(E \setminus K_n) < \varepsilon_n$  for each  $n$ , where  $\varepsilon_n > 0$ , then  $K = \bigcap_{n \in \mathbb{N}} K_n$  is compact, and  $\lambda(E \setminus K) < \sum_{n \in \mathbb{N}} \varepsilon_n$ . In particular, this trick is very useful with  $\varepsilon_n = 2^{-n} \varepsilon$  for some fixed  $\varepsilon > 0$ .

It is tempting to state this as follows: *The intersection of a countable union of compacts, each of which is almost all of  $E$ , is almost all of  $E$ .* But then we should not forget that the differences will add up, it's just that we can arrange for a countable sum of small positive numbers to be small if we wish – and we *do* wish.

## Principle: Measurable functions are almost continuous

This is also known as **Lusin's theorem**. It has many variants. We shall state and prove two of them here.

**2 Theorem (Lusin)** *Let  $f: E \rightarrow \mathbb{C}$  be Lebesgue measurable, with  $E \in \mathcal{M}$  and  $\lambda(E) < \infty$ . Then for any  $\varepsilon > 0$  there is a compact set  $K \subseteq E$  with  $\lambda(E \setminus K) < \varepsilon$  so that  $f|_K$  is continuous.*

**Proof:** We prove it first for simple functions, say  $f = \sum_{k=1}^n a_k \chi_{A_k}$  with  $A_k \in \mathcal{M}$  and  $E = \bigsqcup_{k=1}^n A_k$ . Pick compact  $K_k \subseteq A_k$  with  $\lambda(A_k \setminus K_k) < \varepsilon/n$ , and set  $K = \bigsqcup_{k=1}^n K_k$ . Then  $f$  is constant on each  $K_k$ , and since disjoint compact sets have a mutual positive distance, it follows that  $f|_K$  is continuous. Furthermore  $E \setminus K = \bigsqcup_{k=1}^n (A_k \setminus K_k)$ , so  $\lambda(E \setminus K) < n \cdot (\varepsilon/n) = \varepsilon$ .

Next, we consider a measurable function  $f \geq 0$ . We may allow the value  $f(x) = \infty$ , but must assume  $f(x) < \infty$  a.e. Then, with of Theorem 1 applied to a set  $f^{-1}([0, n])$  for large enough  $n$ , we may find a compact  $L \subseteq E$  with  $\lambda(E \setminus L) < \varepsilon$  so that  $f|_L$  is bounded.

Now recall the proof of the monotone convergence theorem, where we showed that this sequence  $(\phi_n)$  of simple functions converges to  $f$  pointwise:

$$\phi_n(x) = \sum_{k=1}^{n \cdot 2^n} 2^{-n} k \cdot [2^{-n} k \leq f(x) < 2^{-n}(k+1)].$$

Moreover, the convergence will be *uniform* on  $L$ . In fact, when  $n$  is large enough (so  $f \leq n$  on  $L$ ), we find  $|f - \phi_n| \leq 2^{-n}$  on  $L$ . Now we make good use of the remarks after Theorem 1: Let  $K_n \subseteq E$  be compact with  $\lambda(E \setminus K_n) < 2^{-n} \varepsilon$ , and define  $K = L \cap (\bigcap_{n \in \mathbb{N}} K_n)$ . Then  $\lambda(E \setminus K) < 2\varepsilon$ . And since  $\phi_n|_K$  is continuous for each  $n$  and the convergence is uniform on  $K$ , then the limit  $f|_K$  is continuous.

Extending the result first to arbitrary measurable extended real-valued functions  $f$  with  $|f| < \infty$  a.e., and then measurable complex-valued functions, is straightforward. ■

**3 Theorem (Lusin)** *Let  $f$  be either an extended real-valued function on  $\mathbb{R}$  with  $|f| < \infty$  a.e. or a complex-valued function on  $\mathbb{R}$ . Then  $f$  is measurable if, and only if, for each  $\varepsilon > 0$  there is a continuous function  $g$  on  $\mathbb{R}$  and a closed set  $F \subseteq \mathbb{R}$  with  $\lambda(\mathbb{R} \setminus F) < \varepsilon$  so that  $f = g$  on  $F$ .*

**Proof:** We start with the easy part, the “if” direction. So we assume that the given condition holds. Then for each  $n \in \mathbb{N}$  we can find a continuous function  $g_n$  and a closed set  $F_n$  with  $\lambda(\mathbb{R} \setminus F_n) < 2^{-n}$  so that  $f_n = g_n$  on  $F_n$ . Let  $h_n = [F_n] f_n$ . This function is measurable, since it equals  $[F_n] g_n$ , and both  $[F_n]$  and  $g_n$  are measurable. Let  $H = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} F_k$ . Then  $\mathbb{R} \setminus H = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} (\mathbb{R} \setminus F_k)$ , and  $\lambda(\bigcup_{k \geq n} (\mathbb{R} \setminus F_k)) < 2^{1-n}$ , so  $\lambda(\mathbb{R} \setminus H) = 0$ . Also  $h_n \rightarrow f$  pointwise on  $H$ , (in fact

$h_n(x) = f(x)$  for  $n$  large enough if  $x \in H$ ), hence  $f|_H$  is measurable, and it follows that  $f$  is measurable on  $\mathbb{R}$ .

For the converse, assume that  $f$  is measurable. Let  $\varepsilon > 0$ . For each  $n$ , pick a compact set  $K_n \subseteq (-n, n)$  with  $\lambda((-n, n) \setminus K) \leq 2^{-n}\varepsilon$  so that  $f|_{K_n}$  is continuous. Let

$$F = \bigcap_{n \in \mathbb{N}} (K_n \cup (\mathbb{R} \setminus (-n, n))).$$

Then  $F$  is closed, since it is the intersection of closed sets. Also

$$\mathbb{R} \setminus F = \bigcup_{n \in \mathbb{N}} ((-n, n) \setminus K_n),$$

so  $\lambda(\mathbb{R} \setminus F) < \varepsilon$ . Moreover for any  $n$ ,  $F \cap (-n, n) \subseteq K_n$ , so  $f|_F$  is continuous on  $(-n, n)$ . Since continuity is a local property,  $f|_F$  is continuous.

It remains to extend the continuous function  $f|_F$  to a continuous function  $g$  on  $\mathbb{R}$ . The open set  $\mathbb{R} \setminus F$  is a disjoint union of intervals (the components of  $\mathbb{R} \setminus F$ ), whose finite endpoints belong to  $F$ . Extend  $f|_F$  by linear interpolation between the endpoints for any bounded component of  $\mathbb{R} \setminus F$ , and by setting it to a constant (equal to the value at one endpoint) on any unbounded component. The continuity of the function so defined is left as an exercise. ■

### Principle: Pointwise convergence is almost uniform

– but only on sets of *finite measure* – the obvious counterexample otherwise being  $f_n(x) = x/n$  on  $\mathbb{R}$ : Clearly,  $f_n \rightarrow 0$  pointwise, but the convergence is *far* from uniform.

**4 Theorem (Egorov)** *Assume a sequence of measurable functions on a Lebesgue measurable set  $E \subset \mathbb{R}$  with  $\lambda(E) < \infty$  is given. If the sequence converges pointwise with a finite limit a.e. on  $E$ , then for any  $\varepsilon > 0$  there is a compact set  $K$  with  $\lambda(E \setminus K) < \varepsilon$  on which the sequence converges uniformly.*

**Proof:** Since the limit is finite a.e., we can subtract it from the given sequence. In other words, we may assume that the given sequence  $(f_n)$  converges to 0 pointwise a.e. Define

$$g_n(x) = \sup_{k \geq n} f_k(x).$$

Then  $g_n \searrow 0$  a.e., and we only need to find a compact set with small complement on which this convergence is uniform.

To this end, let  $K$  be a compact set with  $\lambda(E \setminus K) < \varepsilon$  so that every  $g_n$  is continuous on  $K$  and  $g_n \searrow 0$  on  $K$ .

The proof is now finished by an appeal to Dini's theorem, which we state and prove below. ■

**5 Theorem (Dini)** *A monotone sequence of continuous functions converging pointwise on a compact set, converges uniformly.*

**Proof:** We may assume, without loss of generality, that the given sequence  $(g_n)$  decreases pointwise to 0 on a compact set  $K$ .

Assume the convergence is *not uniform*. Then there is some  $\varepsilon > 0$  so that for every  $n$ , there is some  $x_n \in K$  with  $g_n(x_n) > \varepsilon$ . (We only need this for infinitely many  $n$ , but the monotonicity of the sequence buys us this much extra.)

Let  $x$  be an accumulation point of the sequence  $(x_n)$ . Since  $g_n(x) \rightarrow 0$ , there is some  $n$  with  $g_n(x) < \varepsilon$ . And since  $g_n$  is continuous, there is some  $\delta > 0$  so that  $g_n(y) < \varepsilon$  whenever  $|x - y| < \delta$ . But if  $k$  is large enough then  $|x - x_k| < \delta$ , and in particular if also  $k \geq n$  we then get  $g_k(x_k) \leq g_n(x_k) < \varepsilon$ , which contradicts the choice of  $x_k$  so  $g_k(x_k) \geq \varepsilon$ .

An *alternative proof* is to note that if the convergence is not uniform, there is some  $\varepsilon > 0$  so that the set  $F_n = \{x \in K : g_n(x) \geq \varepsilon\}$  is nonempty for all  $n$ . But then this is a decreasing sequence of nonempty closed subsets of  $K$ , and hence nonempty by the compactness of  $K$ . If  $x \in \bigcap_{n \in \mathbb{N}} F_n$ , then  $g_n(x) \geq \varepsilon$  for all  $n$ , which contradicts the pointwise convergence. ■