Outer Lebesgue measure

Recall that *outer Lebesgue measure* λ^* has these properties:

- For every interval I, $\lambda^*(I)$ is the length of I, which we also write $\lambda(I)$
- (Subadditivity) For any sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}$,

$$\lambda^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \lambda^*(A_n)$$

Carathéodory's criterion and Lebesgue measurability

We say that a set $E \subseteq \mathbb{R}$ is *Lebesgue measurable* if

$$\lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \setminus E) \qquad \text{for all } W \subseteq \mathbb{R}.$$

The criterion above is known as *Carathéodory's criterion.* It can be applied in more abstract settings than the present.

It is useful to observe that the inequality "$\leq$" holds by subaddtivity, so to prove that a set is Lebesgue measurable, we only need to prove

$$\lambda^*(W) \geq \lambda^*(W \cap E) + \lambda^*(W \setminus E) \qquad \text{for all } W \subseteq \mathbb{R}.$$

We write $\mathcal{M}$ for the set of Lebesgue measurable sets.

Intervals and Lebesgue measurability

Lemma. *A set $E \subseteq \mathbb{R}$ is Lebesgue measurable if and only if*

$$\lambda^*(I) \geq \lambda^*(I \cap E) + \lambda^*(I \setminus E) \qquad \text{for every open interval } I.$$

To prove the nontrivial direction, assume the above condition holds, and let $W \subseteq \mathbb{R}$. Let $(I_n)_{n\in\mathbb{N}}$ be a cover of W by open intervals. We find

$$\begin{aligned}
\lambda^*(W \cap E) + \lambda^*(W \setminus E) &\leq \sum_{n\in\mathbb{N}} \lambda^*(I_n \cap E) + \sum_{n\in\mathbb{N}} \lambda^*(I_n \setminus E) && \text{by subadditivity} \\
&= \sum_{n\in\mathbb{N}} \big(\lambda^*(I_n \cap E) + \lambda^*(I_n \setminus E)\big) && \text{joining the sums} \\
&\leq \sum_{n\in\mathbb{N}} \lambda^*(I_n). && \text{by the assumption}
\end{aligned}$$

Since this holds for *every* cover of W by intervals, it follows that

$$\lambda^*(W \cap E) + \lambda^*(W \setminus E) \leq \lambda^*(W),$$

which shows that E is Lebesgue measurable.

Corollary. *Every interval is Lebesgue measurable.*

A finite additivity result

In this section, we prove finite additivity of outer Lebesgue measure when applied to pairwise disjoint Lebesgue measurable sets $A_1, A_2, \ldots, A_n$:

$$\lambda^*\Big(\bigsqcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \lambda^*(A_k).$$

We shall, however, need a slightly more general version of this result.

For any pairwise disjoint sets $A_1, A_2, \ldots, A_n$ with $A_1, A_2, \ldots, A_{n-1}$ Lebesgue measurable and any $W \subseteq \mathbb{R}$, we have

$$\lambda^*\Big(W \cap \bigsqcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \lambda^*(W \cap A_k).$$

A word on notation: I use the notation $\bigsqcup$ to indicate a *disjoint union*, i.e., a union of pairwise disjoint sets. We use this symbol, and its binary variant $\sqcup$, for the union *only* when the sets involved are known (or assumed) to be pairwise disjoint. So $A \sqcup B = A \cup B$ if $A \cap B = \emptyset$, but the notation $A \sqcup B$ should be considered to be meaningless otherwise.

Proof: We prove it first for $n = 2$: If A, B are pairwise disjoint and A is Lebesgue measurable, then

$$\lambda^*(W \cap (A \sqcup B)) = \lambda^*(W \cap A) + \lambda^*(W \cap B),$$

which follows directly from the measurability of A because $(W \cap (A \sqcup B)) \cap A = W \cap A$ and $(W \cap (A \sqcup B)) \setminus A = W \cap B$.

In the general case, using the above with $A = A_1$ and $B = \bigsqcup_{k=2}^{n} A_k$ yields

$$\lambda^*\Big(W \cap \bigsqcup_{k=1}^{n} A_k\Big) = \lambda^*(W \cap A_1) + \lambda^*\Big(W \cap \bigsqcup_{k=2}^{n} A_k\Big),$$

and applying this inductively to the last term yields the desired result.

***Claim*: The Lebesgue measurable sets form an algebra of sets.**

It follows directly from the definition that $\mathcal{M}$ is closed under complements.

We next show that it is closed under finite unions. So let $A, B \in \mathcal{M}$. For any $W \subseteq \mathbb{R}$ we find

$$
\begin{aligned}
\lambda^*(W) &= \lambda^*(W \cap A) + \lambda^*(W \cap A^c) \\
&= \overbrace{\lambda^*(W \cap A \cap B) + \lambda^*(W \cap A \cap B^c)} + \overbrace{\lambda^*(W \cap A^c \cap B)} + \lambda^*(W \cap A^c \cap B^c) \\
&\ge \lambda^*(W \cap (A \cup B)) + \lambda^*(W \cap (A^c \cap B^c)) \\
&= \lambda^*(W \cap (A \cup B)) + \lambda^*(W \setminus (A \cup B))
\end{aligned}
$$

note: $(A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) = A \cup B$

(using subadditivity in the third line), which shows that $A \cup B \in \mathcal{M}$. Our claim follows from this.

A countable additivity result

For any sequence $(A_k)_{k\in\mathbb{N}}$ of pairwise disjoint Lebesgue measurable sets and any $W \subseteq \mathbb{R}$, we have

$$\lambda^*\left(W \cap \bigsqcup_{k\in\mathbb{N}} A_k\right) = \sum_{k\in\mathbb{N}} \lambda^*(W \cap A_k).$$

Proof: We use our finite additivity result, noting that we do not yet know, nor do we need to know, that $\bigsqcup_{k=n+1}^{\infty} A_k$ is measurable:

$$\begin{aligned}
\lambda^*\left(W \cap \bigsqcup_{k\in\mathbb{N}} A_k\right) &= \lambda^*\left(\left(W \cap \bigsqcup_{k=1}^{n} A_k\right) \sqcup \left(W \cap \bigsqcup_{k=n+1}^{\infty} A_k\right)\right) \\
&= \sum_{k=1}^{n} \lambda^*(W \cap A_k) + \lambda^*\left(\left(W \cap \bigsqcup_{k=n+1}^{\infty} A_k\right)\right) \\
&\geq \sum_{k=1}^{n} \lambda^*(W \cap A_k),
\end{aligned}$$

and letting $n \to \infty$ we conclude that

$$\lambda^*\left(W \cap \bigsqcup_{k=1}^{n} A_k\right) \geq \lambda^*(W \cap A_1) + \lambda^*\left(W \cap \bigsqcup_{k=2}^{n} A_k\right).$$

Since the opposite inequality holds by countable subadditivity, the proof is complete.

***Claim*: The Lebesgue measurable sets form a σ-algebra.**

To prove this, it only remains to show that a countable union of Lebesgue measurable sets is Lebesgue measurable.

So let $(E_n)_{n\in\mathbb{N}}$ be a sequence of sets, $E_n \in \mathcal{M}$, and write

$$E = \bigcup_{k\in\mathbb{N}} E_k.$$

Furthermore, let

$$B_n = \bigcup_{k=1}^{n} E_k, \quad B_0 = \emptyset, \quad A_n = B_n \setminus B_{n-1}, \quad \text{so that } B_n = \bigsqcup_{k=1}^{n} A_k \text{ and } E = \bigsqcup_{k=1}^{\infty} A_k$$

and note that all the sets A_n, B_n are Lebesgue measurable.

Now let $W \subseteq \mathbb{R}$ be any set. Then

$$\begin{aligned} \lambda^*(W) &= \lambda^*(W\cap B_n) + \lambda^*(W\setminus B_n) && \text{B_n is measurable} \\ &\geq \lambda^*(W\cap B_n) + \lambda^*(W\setminus E) && \text{$B_n \subseteq E$, so $W\setminus B_n \supseteq W\setminus E$} \end{aligned}$$

Now let $n\to\infty$:

$$\begin{aligned} \lambda^*(W) &\geq \lim_{n\to\infty} \lambda^*(W\cap B_n) + \lambda^*(W\setminus E) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \lambda^*(W\cap A_n) + \lambda^*(W\setminus E) && \text{by finite additivity} \\ &= \sum_{k=1}^{\infty} \lambda^*(W\cap A_n) + \lambda^*(W\setminus E) \\ &= \lambda^*(W\cap E) + \lambda^*(W\setminus E) && \text{by countable additivity,} \end{aligned}$$

showing that $E \in \mathcal{M}$ as claimed.

We *really* only needed countable *sub*additivity in the last line to obtain the needed inequality.

Corollary. *Every open or closed set is Lebesgue measurable, and so is every Borel set.*

Proof. Any open set is a countable union of open intervals, and hence measurable. Any closed set is the complement of an open set, and hence measurable. And the set of Borel sets is the σ-algebra generated by the open subsets of $\mathbb{R}$. Since $\mathcal{M}$ is a σ-algebra containing the open subsets of $\mathbb{R}$, it contains all Borel sets.

Summary – Lebesgue measure

We define *Lebesgue measure* λ to be the restriction of Lebesgue outer measure λ^* to the Lebesgue measurable sets:

$$\lambda(E) = \lambda^*(E), \qquad E \in \mathcal{M}.$$

It is countably additive:

$$\lambda\Big(\bigcup_{k\in\mathbb{N}} A_k\Big) = \sum_{k\in\mathbb{N}} \lambda(A_k)$$

where $(A_k)_{k\in\mathbb{N}}$ is a sequence of pairwise disjoint Lebesgue measurable sets.