

## Outer Lebesgue measure

Recall that *outer Lebesgue measure*  $\lambda^*$  has these properties:

- For every interval  $I$ ,  $\lambda^*(I)$  is the length of  $I$ , which we also write  $\lambda(I)$
- (Subadditivity) For any sequence  $(A_n)_{n \in \mathbb{N}}$  of subsets of  $\mathbb{R}$ ,

$$\lambda^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n)$$

## Carathéodory's criterion and Lebesgue measurability

We say that a set  $E \subseteq \mathbb{R}$  is *Lebesgue measurable* if

$$\lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \setminus E) \quad \text{for all } W \subseteq \mathbb{R}.$$

The criterion above is known as *Carathéodory's criterion*. It can be applied in more abstract settings than the present.

It is useful to observe that the inequality “ $\leq$ ” holds by subadditivity, so to prove that a set is Lebesgue measurable, we only need to prove

$$\lambda^*(W) \geq \lambda^*(W \cap E) + \lambda^*(W \setminus E) \quad \text{for all } W \subseteq \mathbb{R}.$$

We write  $\mathcal{M}$  for the set of Lebesgue measurable sets.

## Intervals and Lebesgue measurability

**Lemma.** *A set  $E \subseteq \mathbb{R}$  is Lebesgue measurable if and only if*

$$\lambda^*(I) \geq \lambda^*(I \cap E) + \lambda^*(I \setminus E) \quad \text{for every open interval } I.$$

To prove the nontrivial direction, assume the above condition holds, and let  $W \subseteq \mathbb{R}$ . Let  $(I_n)_{n \in \mathbb{N}}$  be a cover of  $W$  by open intervals. We find

$$\begin{aligned} \lambda^*(W \cap E) + \lambda^*(W \setminus E) &\leq \sum_{n \in \mathbb{N}} \lambda^*(I_n \cap E) + \sum_{n \in \mathbb{N}} \lambda^*(I_n \setminus E) && \text{by subadditivity} \\ &= \sum_{n \in \mathbb{N}} (\lambda^*(I_n \cap E) + \lambda^*(I_n \setminus E)) && \text{joining the sums} \\ &\leq \sum_{n \in \mathbb{N}} \lambda^*(I_n). && \text{by the assumption} \end{aligned}$$

Since this holds for *every* cover of  $W$  by intervals, it follows that

$$\lambda^*(W \cap E) + \lambda^*(W \setminus E) \leq \lambda^*(W),$$

which shows that  $E$  is Lebesgue measurable.

**Corollary.** *Every interval is Lebesgue measurable.*

## A finite additivity result

In this section, we prove finite additivity of outer Lebesgue measure when applied to pairwise disjoint Lebesgue measurable sets  $A_1, A_2, \dots, A_n$ :

$$\lambda^* \left( \bigsqcup_{k=1}^n A_k \right) = \sum_{k=1}^n \lambda^*(A_k).$$

We shall, however, need a slightly more general version of this result.

*For any pairwise disjoint Lebesgue measurable sets  $A_1, A_2, \dots, A_n$  and any  $W \subseteq \mathbb{R}$ , we have*

$$\lambda^* \left( W \cap \bigsqcup_{k=1}^n A_k \right) = \sum_{k=1}^n \lambda^*(W \cap A_k).$$

A word on notation: I use the notation  $\bigsqcup$  to indicate a *disjoint union*, i.e., a union of pairwise disjoint sets. We use this symbol, and its binary variant  $\sqcup$ , for the union *only* when the sets involved are known (or assumed) to be pairwise disjoint. So  $A \sqcup B = A \cup B$  if  $A \cap B = \emptyset$ , but the notation  $A \sqcup B$  should be considered to be meaningless otherwise.

*Proof:* We note first a slightly stronger statement for  $n = 2$ : If  $A, B$  are pairwise disjoint and  $A$  is Lebesgue measurable, then

$$\lambda^*(W \cap (A \sqcup B)) = \lambda^*(W \cap A) + \lambda^*(W \cap B),$$

which follows directly from the measurability of  $A$  because  $(W \cap (A \sqcup B)) \cap A = W \cap A$  and  $(W \cap (A \sqcup B)) \setminus A = W \cap B$ .

In the general case, using the above with  $A = A_1$  and  $B = \bigsqcup_{k=2}^n A_k$  yields

$$\lambda^* \left( W \cap \bigsqcup_{k=1}^n A_k \right) = \lambda^*(W \cap A_1) + \lambda^* \left( W \cap \bigsqcup_{k=2}^n A_k \right),$$

and applying this inductively to the last term yields the desired result.

**Claim: The Lebesgue measurable sets form an algebra of sets.**

It follows directly from the definition that  $\mathcal{M}$  is closed under complements.

We next show that it is closed under finite unions. So let  $A, B \in \mathcal{M}$ . For any  $W \subseteq \mathbb{R}$  we find

$$\begin{aligned}\lambda^*(W) &= \lambda^*(W \cap A) + \lambda^*(W \cap A^c) \\ &= \underbrace{\lambda^*(W \cap A \cap B) + \lambda^*(W \cap A \cap B^c)}_{\text{note: } (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) = A \cup B} + \underbrace{\lambda^*(W \cap A^c \cap B) + \lambda^*(W \cap A^c \cap B^c)} \\ &\geq \lambda^*(W \cap (A \cup B)) + \lambda^*(W \cap (A^c \cap B^c)) \\ &= \lambda^*(W \cap (A \cup B)) + \lambda^*(W \setminus (A \cup B))\end{aligned}$$

(using subadditivity in the third line), which shows that  $A \cup B \in \mathcal{M}$ . Our claim follows from this.

**Claim: The Lebesgue measurable sets form a  $\sigma$ -algebra.**

To prove this, it only remains to show that a countable union of Lebesgue measurable sets is Lebesgue measurable. So let  $(E_n)_{n \in \mathbb{N}}$  be a sequence of sets,  $E_n \in \mathcal{M}$ , and write

$$E = \bigcup_{k \in \mathbb{N}} E_k.$$

Furthermore, let

$$B_n = \bigcup_{k=1}^n E_k, \quad B_0 = \emptyset, \quad A_n = B_n \setminus B_{n-1}, \quad \text{so that } B_n = \bigsqcup_{k=1}^n A_k \text{ and } E = \bigsqcup_{k=1}^{\infty} A_k$$

and note that all the sets  $A_n, B_n$  are Lebesgue measurable.

Now let  $W \subseteq \mathbb{R}$  be any set. I *claim*:

$$\underbrace{\lambda^*(W \cap E)}_a = \underbrace{\sum_{k=1}^{\infty} \lambda^*(W \cap A_k)}_b = \underbrace{\lim_{n \rightarrow \infty} \lambda^*(W \cap B_n)}_c.$$

The second equality ( $b = c$ ) follows direct from the finite additivity results shown earlier, i.e.,

$$\lambda^*(W \cap B_n) = \sum_{k=1}^n \lambda^*(W \cap A_k),$$

and then taking the limit as  $n \rightarrow \infty$ . The inequality  $a \leq b$  is just the countable subadditivity of  $\lambda^*$ . The inequality  $c \leq a$  follows from  $B_n \subseteq E$ , so that  $\lambda^*(W \cap B_n) \leq \lambda^*(W \cap E)$ . Thus the claim is proved.

Now we use the measurability of  $B_n$ :

$$\lambda^*(W) = \lambda^*(W \cap B_n) + \lambda^*(W \setminus B_n) \geq \lambda^*(W \cap B_n) + \lambda^*(W \setminus E)$$

because  $B_n \subseteq E$ , so  $W \setminus B_n \supseteq W \setminus E$ . Therefore, taking the limit and using the above claim we get

$$\lambda^*(W) \geq \lim_{n \rightarrow \infty} \lambda^*(W \cap B_n) + \lambda^*(W \setminus E) = \lambda^*(W \cap E) + \lambda^*(W \setminus E),$$

showing that  $E \in \mathcal{M}$  as claimed.

**Corollary.** *Every open or closed set is Lebesgue measurable, and so is every Borel set.*

*Proof.* Any open set is a countable union of open intervals, and hence measurable. Any closed set is the complement of an open set, and hence measurable. And the set of Borel sets is the  $\sigma$ -algebra generated by the open subsets of  $\mathbb{R}$ . Since  $\mathcal{M}$  is a  $\sigma$ -algebra containing the open subsets of  $\mathbb{R}$ , it contains all Borel sets.

## Countable additivity of the Lebesgue measure

As a bonus from the above proof that  $\mathcal{M}$  is a  $\sigma$ -algebra, we have

$$\lambda^*\left(W \cap \bigsqcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \lambda^*(W \cap A_n)$$

whenever  $(A_n)_{n \in \mathbb{N}}$  is a sequence of pairwise disjoint Lebesgue measurable sets.

We define *Lebesgue measure*  $\lambda$  to be the restriction of Lebesgue outer measure  $\lambda^*$  to the Lebesgue measurable sets:

$$\lambda(E) = \lambda^*(E), \quad E \in \mathcal{M}.$$

The countable additivity of  $\lambda$  is an immediate special case of the equality above, with  $W = \mathbb{R}$ .