

## Outer Lebesgue measure

Recall that *outer Lebesgue measure*  $\lambda^*$  has these properties:

- For every interval  $I$ ,  $\lambda^*(I)$  is the length of  $I$ , which we also write  $\lambda(I)$
- (Subadditivity) For any sequence  $(A_n)_{n \in \mathbb{N}}$  of subsets of  $\mathbb{R}$ ,

$$\lambda^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n)$$

## Carathéodory's criterion and Lebesgue measurability

We say that a set  $E \subseteq \mathbb{R}$  is *Lebesgue measurable* if

$$\lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \setminus E) \quad \text{for all } W \subseteq \mathbb{R}.$$

The criterion above is known as *Carathéodory's criterion*. It can be applied in more abstract settings than the present.

It is useful to observe that the inequality “ $\leq$ ” holds by subadditivity, so to prove that a set is Lebesgue measurable, we only need to prove

$$\lambda^*(W) \geq \lambda^*(W \cap E) + \lambda^*(W \setminus E) \quad \text{for all } W \subseteq \mathbb{R}.$$

We write  $\mathcal{M}$  for the set of Lebesgue measurable sets.

## Intervals and Lebesgue measurability

**Lemma.** *A set  $E \subseteq \mathbb{R}$  is Lebesgue measurable if and only if*

$$\lambda^*(I) \geq \lambda^*(I \cap E) + \lambda^*(I \setminus E) \quad \text{for every open interval } I.$$

To prove the nontrivial direction, assume the above condition holds, and let  $W \subseteq \mathbb{R}$ . Let  $(I_n)_{n \in \mathbb{N}}$  be a cover of  $W$  by open intervals. We find

$$\begin{aligned} \lambda^*(W \cap E) + \lambda^*(W \setminus E) &\leq \sum_{n \in \mathbb{N}} \lambda^*(I_n \cap E) + \sum_{n \in \mathbb{N}} \lambda^*(I_n \setminus E) && \text{by subadditivity} \\ &= \sum_{n \in \mathbb{N}} (\lambda^*(I_n \cap E) + \lambda^*(I_n \setminus E)) && \text{joining the sums} \\ &\leq \sum_{n \in \mathbb{N}} \lambda^*(I_n). && \text{by the assumption} \end{aligned}$$

Since this holds for *every* cover of  $W$  by intervals, it follows that

$$\lambda^*(W \cap E) + \lambda^*(W \setminus E) \leq \lambda^*(W),$$

which shows that  $E$  is Lebesgue measurable.

**Corollary.** *Every interval is Lebesgue measurable.*

## A finite additivity result

In this section, we prove finite additivity of outer Lebesgue measure when applied to pairwise disjoint Lebesgue measurable sets  $A_1, A_2, \dots, A_n$ :

$$\lambda^* \left( \bigsqcup_{k=1}^n A_k \right) = \sum_{k=1}^n \lambda^*(A_k).$$

We shall, however, need a slightly more general version of this result.

*For any pairwise disjoint sets  $A_1, A_2, \dots, A_n$  with  $A_1, A_2, \dots, A_{n-1}$  Lebesgue measurable and any  $W \subseteq \mathbb{R}$ , we have*

$$\lambda^* \left( W \cap \bigsqcup_{k=1}^n A_k \right) = \sum_{k=1}^n \lambda^*(W \cap A_k).$$

A word on notation: I use the notation  $\bigsqcup$  to indicate a *disjoint union*, i.e., a union of pairwise disjoint sets. We use this symbol, and its binary variant  $\sqcup$ , for the union *only* when the sets involved are known (or assumed) to be pairwise disjoint. So  $A \sqcup B = A \cup B$  if  $A \cap B = \emptyset$ , but the notation  $A \sqcup B$  should be considered to be meaningless otherwise.

*Proof:* We prove it first for  $n = 2$ : If  $A, B$  are pairwise disjoint and  $A$  is Lebesgue measurable, then

$$\lambda^*(W \cap (A \sqcup B)) = \lambda^*(W \cap A) + \lambda^*(W \cap B),$$

which follows directly from the measurability of  $A$  because  $(W \cap (A \sqcup B)) \cap A = W \cap A$  and  $(W \cap (A \sqcup B)) \setminus A = W \cap B$ .

In the general case, using the above with  $A = A_1$  and  $B = \bigsqcup_{k=2}^n A_k$  yields

$$\lambda^* \left( W \cap \bigsqcup_{k=1}^n A_k \right) = \lambda^*(W \cap A_1) + \lambda^* \left( W \cap \bigsqcup_{k=2}^n A_k \right),$$

and applying this inductively to the last term yields the desired result.

**Claim: The Lebesgue measurable sets form an algebra of sets.**

It follows directly from the definition that  $\mathcal{M}$  is closed under complements.

We next show that it is closed under finite unions. So let  $A, B \in \mathcal{M}$ . For any  $W \subseteq \mathbb{R}$  we find

$$\begin{aligned}\lambda^*(W) &= \lambda^*(W \cap A) + \lambda^*(W \cap A^c) \\ &= \underbrace{\lambda^*(W \cap A \cap B) + \lambda^*(W \cap A \cap B^c)}_{\text{note: } (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) = A \cup B} + \underbrace{\lambda^*(W \cap A^c \cap B) + \lambda^*(W \cap A^c \cap B^c)} \\ &\geq \lambda^*(W \cap (A \cup B)) + \lambda^*(W \cap (A^c \cap B^c)) \\ &= \lambda^*(W \cap (A \cup B)) + \lambda^*(W \setminus (A \cup B))\end{aligned}$$

(using subadditivity in the third line), which shows that  $A \cup B \in \mathcal{M}$ . Our claim follows from this.

## A countable additivity result

For any sequence  $(A_k)_{k \in \mathbb{N}}$  of pairwise disjoint Lebesgue measurable sets and any  $W \subseteq \mathbb{R}$ , we have

$$\lambda^* \left( W \cap \bigsqcup_{k \in \mathbb{N}} A_k \right) = \sum_{k \in \mathbb{N}} \lambda^* (W \cap A_k).$$

*Proof:* We use our finite additivity result, noting that we do not yet know, nor do we need to know, that  $\bigsqcup_{k=n+1}^{\infty} A_k$  is measurable:

$$\begin{aligned} \lambda^* \left( W \cap \bigsqcup_{k \in \mathbb{N}} A_k \right) &= \lambda^* \left( \left( W \cap \bigsqcup_{k=1}^n A_k \right) \sqcup \left( W \cap \bigsqcup_{k=n+1}^{\infty} A_k \right) \right) \\ &= \sum_{k=1}^n \lambda^* (W \cap A_k) + \lambda^* \left( W \cap \bigsqcup_{k=n+1}^{\infty} A_k \right) \\ &\geq \sum_{k=1}^n \lambda^* (W \cap A_k), \end{aligned}$$

and letting  $n \rightarrow \infty$  we conclude that

$$\lambda^* \left( W \cap \bigsqcup_{k \in \mathbb{N}} A_k \right) \geq \sum_{k=1}^{\infty} \lambda^* (W \cap A_k).$$

Since the opposite inequality holds by countable subadditivity, the proof is complete.

**Claim: The Lebesgue measurable sets form a  $\sigma$ -algebra.**

To prove this, it only remains to show that a countable union of Lebesgue measurable sets is Lebesgue measurable.

So let  $(E_n)_{n \in \mathbb{N}}$  be a sequence of sets,  $E_n \in \mathcal{M}$ , and write

$$E = \bigcup_{k \in \mathbb{N}} E_k.$$

Furthermore, let

$$B_n = \bigcup_{k=1}^n E_k, \quad B_0 = \emptyset, \quad A_n = B_n \setminus B_{n-1}, \quad \text{so that } B_n = \bigsqcup_{k=1}^n A_k \text{ and } E = \bigsqcup_{k=1}^{\infty} A_k$$

and note that all the sets  $A_n, B_n$  are Lebesgue measurable.

Now let  $W \subseteq \mathbb{R}$  be any set. Then

$$\begin{aligned} \lambda^*(W) &= \lambda^*(W \cap B_n) + \lambda^*(W \setminus B_n) && B_n \text{ is measurable} \\ &\geq \lambda^*(W \cap B_n) + \lambda^*(W \setminus E) && B_n \subseteq E, \text{ so } W \setminus B_n \supseteq W \setminus E \end{aligned}$$

Now let  $n \rightarrow \infty$ :

$$\begin{aligned} \lambda^*(W) &\geq \lim_{n \rightarrow \infty} \lambda^*(W \cap B_n) + \lambda^*(W \setminus E) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda^*(W \cap A_k) + \lambda^*(W \setminus E) && \text{by finite additivity} \\ &= \sum_{k=1}^{\infty} \lambda^*(W \cap A_k) + \lambda^*(W \setminus E) \\ &= \lambda^*(W \cap E) + \lambda^*(W \setminus E) && \text{by countable additivity,} \end{aligned}$$

showing that  $E \in \mathcal{M}$  as claimed.

We *really* only needed countable *sub*additivity in the last line to obtain the needed inequality.

**Corollary.** *Every open or closed set is Lebesgue measurable, and so is every Borel set.*

*Proof.* Any open set is a countable union of open intervals, and hence measurable. Any closed set is the complement of an open set, and hence measurable. And the set of Borel sets is the  $\sigma$ -algebra generated by the open subsets of  $\mathbb{R}$ . Since  $\mathcal{M}$  is a  $\sigma$ -algebra containing the open subsets of  $\mathbb{R}$ , it contains all Borel sets.

## Summary – Lebesgue measure

We define *Lebesgue measure*  $\lambda$  to be the restriction of Lebesgue outer measure  $\lambda^*$  to the Lebesgue measurable sets:

$$\lambda(E) = \lambda^*(E), \quad E \in \mathcal{M}.$$

It is countably additive:

$$\lambda\left(\bigcup_{k \in \mathbb{N}} A_k\right) = \sum_{k \in \mathbb{N}} \lambda(A_k)$$

where  $(A_k)_{k \in \mathbb{N}}$  is a sequence of pairwise disjoint Lebesgue measurable sets.