Outer Lebesgue measure

Recall that *outer Lebesgue measure* λ^* has these properties:

- For every interval I, $\lambda^*(I)$ is the length of I, which we also write $\lambda(I)$
- (Subadditivity) For any sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}$,

$$\lambda^*\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \le \sum_{n\in\mathbb{N}} \lambda^*(A_n)$$

Carathéodory's criterion and Lebesgue measurability

We say that a set $E \subseteq \mathbb{R}$ is *Lebesgue measurable* if

$$\lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \setminus E) \qquad \text{for all } W \subseteq \mathbb{R}.$$

The criterion above is known as *Carathéodory's criterion.* It can be applied in more abstract settings than the present.

It is useful to observe that the inequality "≤" holds by subaddtivity, so to prove that a set is Lebesgue measurable, we only need to prove

$$\lambda^*(W) \ge \lambda^*(W \cap E) + \lambda^*(W \setminus E) \qquad \text{for all } W \subseteq \mathbb{R}.$$

We write $\mathcal{M}$ for the set of Lebesgue measurable sets.

Intervals and Lebesgue measurability

Lemma. *A set $E \subseteq \mathbb{R}$ is Lebesgue measurable if and only if*

$$\lambda^*(I) \ge \lambda^*(I \cap E) + \lambda^*(I \setminus E) \qquad \textit{for every open interval } I.$$

To prove the nontrivial direction, assume the above condition holds, and let $W \subseteq \mathbb{R}$. Let $(I_n)_{n\in\mathbb{N}}$ be a cover of W by open intervals. We find

$$\begin{aligned}
\lambda^*(W \cap E) + \lambda^*(W \setminus E) &\le \sum_{n\in\mathbb{N}} \lambda^*(I_n \cap E) + \sum_{n\in\mathbb{N}} \lambda^*(I_n \setminus E) && \text{by subadditivity}\\
&= \sum_{n\in\mathbb{N}} \big(\lambda^*(I_n \cap E) + \lambda^*(I_n \setminus E)\big) && \text{joining the sums}\\
&\le \sum_{n\in\mathbb{N}} \lambda^*(I_n). && \text{by the assumption}
\end{aligned}$$

Since this holds for *every* cover of W by intervals, it follows that

$$\lambda^*(W \cap E) + \lambda^*(W \setminus E) \le \lambda^*(W),$$

which shows that E is Lebesgue measurable.

Corollary. *Every interval is Lebesgue measurable.*

A finite additivity result

In this section, we prove finite additivity of outer Lebesgue measure when applied to pairwise disjoint Lebesgue measurable sets $A_1, A_2, \ldots, A_n$:

$$\lambda^*\Big(\bigsqcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \lambda^*(A_k).$$

We shall, however, need a slightly more general version of this result.

For any pairwise disjoint Lebesgue measurable sets $A_1, A_2, \ldots, A_n$ and any $W \subseteq \mathbb{R}$, we have

$$\lambda^*\Big(W \cap \bigsqcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \lambda^*(W \cap A_k).$$

A word on notation: I use the notation $\bigsqcup$ to indicate a *disjoint union*, i.e., a union of pairwise disjoint sets. We use this symbol, and its binary variant $\sqcup$, for the union *only* when the sets involved are known (or assumed) to be pairwise disjoint. So $A \sqcup B = A \cup B$ if $A \cap B = \emptyset$, but the notation $A \sqcup B$ should be considered to be meaningless otherwise.

Proof: We note first a slightly stronger statement for $n = 2$: If A, B are pairwise disjoint and A is Lebesgue measurable, then

$$\lambda^*(W \cap (A \sqcup B)) = \lambda^*(W \cap A) + \lambda^*(W \cap B),$$

which follows directly from the measurability of A because $(W \cap (A \sqcup B)) \cap A = W \cap A$ and $(W \cap (A \sqcup B)) \setminus A = W \cap B$.

In the general case, using the above with $A = A_1$ and $B = \bigsqcup_{k=2}^{n} A_k$ yields

$$\lambda^*\Big(W \cap \bigsqcup_{k=1}^{n} A_k\Big) = \lambda^*(W \cap A_1) + \lambda^*\Big(W \cap \bigsqcup_{k=2}^{n} A_k\Big),$$

and applying this inductively to the last term yields the desired result.

***Claim*: The Lebesgue measurable sets form an algebra of sets.**

It follows directly from the definition that $\mathcal{M}$ is closed under complements.

We next show that it is closed under finite unions. So let $A, B \in \mathcal{M}$. For any $W \subseteq \mathbb{R}$ we find

$$\begin{aligned}
\lambda^*(W) &= \lambda^*(W \cap A) + \lambda^*(W \cap A^c) \\
&= \underbrace{\overbrace{\lambda^*(W \cap A \cap B) + \lambda^*(W \cap A \cap B^c)} + \overbrace{\lambda^*(W \cap A^c \cap B)}}_{\text{note: } (A\cap B)\cup(A\cap B^c)\cup(A^c\cap B) = A\cup B} \overbrace{{}+ \lambda^*(W \cap A^c \cap B^c)} \\
&\geq \lambda^*(W \cap (A \cup B)) + \lambda^*(W \cap (A^c \cap B^c)) \\
&= \lambda^*(W \cap (A \cup B)) + \lambda^*(W \setminus (A \cup B))
\end{aligned}$$

(using subadditivity in the third line), which shows that $A \cup B \in \mathcal{M}$. Our claim follows from this.

Claim: The Lebesgue measurable sets form a σ-algebra.

To prove this, it only remains to show that a countable union of Lebesgue measurable sets is Lebesgue measurable. So let $(E_n)_{n\in\mathbb{N}}$ be a sequence of sets, $E_n \in \mathcal{M}$, and write

$$E = \bigcup_{k\in\mathbb{N}} E_k.$$

Furthermore, let

$$B_n = \bigcup_{k=1}^{n} E_k, \quad B_0 = \emptyset, \quad A_n = B_n \setminus B_{n-1}, \quad \text{so that } B_n = \bigsqcup_{k=1}^{n} A_k \text{ and } E = \bigsqcup_{k=1}^{\infty} A_k$$

and note that all the sets A_n, B_n are Lebesgue measurable.

Now let $W \subseteq \mathbb{R}$ be any set. I *claim*:

$$\underbrace{\lambda^*\Big(W \cap E\Big)}_{a} = \underbrace{\sum_{k=1}^{\infty} \lambda^*(W \cap A_k)}_{b} = \underbrace{\lim_{n\to\infty} \lambda^*(W \cap B_n)}_{c}.$$

The second equality ($b = c$) follows direct from the finite additivity results shown earlier, i.e.,

$$\lambda^*(W \cap B_n) = \sum_{k=1}^{n} \lambda^*(W \cap A_k),$$

and then taking the limit as $n \to \infty$. The inequality $a \le b$ is just the countable subadditivity of λ^*. The inequality $c \le a$ follows from $B_n \subseteq E$, so that $\lambda^*(W \cap B_n) \le \lambda^*(W \cap E)$. Thus the claim is proved.

Now we use the measurability of B_n:

$$\lambda^*(W) = \lambda^*(W \cap B_n) + \lambda^*(W \setminus B_n) \ge \lambda^*(W \cap B_n) + \lambda^*(W \setminus E)$$

because $B_n \subseteq E$, so $W \setminus B_n \supseteq W \setminus E$. Therefore, taking the limit and using the above claim we get

$$\lambda^*(W) \ge \lim_{n\to\infty} \lambda^*(W \cap B_n) + \lambda^*(W \setminus E) = \lambda^*(W \cap E) + \lambda^*(W \setminus E),$$

showing that $E \in \mathcal{M}$ as claimed.

Corollary. *Every open or closed set is Lebesgue measurable, and so is every Borel set.*
Proof. Any open set is a countable union of open intervals, and hence measurable. Any closed set is the complement of an open set, and hence measurable. And the set of Borel sets is the σ-algebra generated by the open subsets of $\mathbb{R}$. Since $\mathcal{M}$ is a σ-algebra containing the open subsets of $\mathbb{R}$, it contains all Borel sets.

Countable additivity of the Lebesgue measure

As a bonus from the above proof that $\mathcal{M}$ is a σ-algebra, we have

$$\lambda^*\Big(W \cap \bigsqcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \lambda^*(W \cap A_n)$$

whenever $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint Lebesgue measurable sets.

We define *Lebesgue measure* λ to be the restriction of Lebesgue outer measure λ^* to the Lebesgue measurable sets:

$$\lambda(E) = \lambda^*(E), \qquad E \in \mathcal{M}.$$

The countable additivity of λ is an immediate special case of the equality above, with $W = \mathbb{R}$.