Outer Lebesgue measure

Recall that *outer Lebesgue measure* λ^* has these properties:

- For every interval I, $\lambda^*(I)$ is the length of I, which we also write $\lambda(I)$
- (Subadditivity) For any sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}$,

$$\lambda^*\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \le \sum_{n\in\mathbb{N}} \lambda^*(A_n)$$

Carathéodory's criterion and Lebesgue measurability

We say that a set $E \subseteq \mathbb{R}$ is *Lebesgue measurable* if

$$\lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \setminus E) \qquad \text{for all } W \subseteq \mathbb{R}.$$

The criterion above is known as *Carathéodory's criterion.* It can be applied in more abstract settings than the present.

It is useful to observe that the inequality "$\le$" holds by subaddtivity, so to prove that a set is Lebesgue measurable, we only need to prove

$$\lambda^*(W) \ge \lambda^*(W \cap E) + \lambda^*(W \setminus E) \qquad \text{for all } W \subseteq \mathbb{R}.$$

We write $\mathcal{M}$ for the set of Lebesgue measurable sets.

Intervals and Lebesgue measurability

Lemma. *A set $E \subseteq \mathbb{R}$ is Lebesgue measurable if and only if*

$$\lambda^*(I) \ge \lambda^*(I \cap E) + \lambda^*(I \setminus E) \qquad \textit{for every open interval } I.$$

To prove the nontrivial direction, assume the above condition holds, and let $W \subseteq \mathbb{R}$. Let $(I_n)_{n\in\mathbb{N}}$ be a cover of W by open intervals. We find

$$\begin{aligned}
\lambda^*(W \cap E) + \lambda^*(W \setminus E) &\le \sum_{n\in\mathbb{N}} \lambda^*(I_n \cap E) + \sum_{n\in\mathbb{N}} \lambda^*(I_n \setminus E) && \text{by subadditivity} \\
&= \sum_{n\in\mathbb{N}} \big(\lambda^*(I_n \cap E) + \lambda^*(I_n \setminus E)\big) && \text{joining the sums} \\
&\le \sum_{n\in\mathbb{N}} \lambda^*(I_n). && \text{by the assumption}
\end{aligned}$$

Since this holds for *every* cover of W by intervals, it follows that

$$\lambda^*(W \cap E) + \lambda^*(W \setminus E) \le \lambda^*(W),$$

which shows that E is Lebesgue measurable.

Corollary. *Every interval is Lebesgue measurable.*

A finite additivity result

For any pairwise disjoint Lebesgue measurable sets $A_1, A_2, \ldots, A_n$ and any $W \subseteq \mathbb{R}$, we have

$$\lambda^*\Big(W \cap \bigsqcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \lambda^*(W \cap A_k).$$

A word on notation: I use the notation $\bigsqcup$ to indicate a *disjoint union*, i.e., a union of pairwise disjoint sets. We use this symbol, and its binary variant $\sqcup$, for the union *only* when the sets involved are known (or assumed) to be pairwise disjoint. So $A \sqcup B = A \cup B$ if $A \cap B = \emptyset$, but the notation $A \sqcup B$ should be considered to be meaningless otherwise.

Proof: Neither side of the equality changes if we replace W by $W \cap \bigsqcup_{k=1}^{n} A_k$, so we might as well assume that$W \subseteq \bigsqcup_{k=1}^{n} A_k$. Now we find

$$\begin{aligned}
\lambda^*(W) &= \lambda^*(W \cap A_1) + \lambda^*(W \setminus A_1) \\
&= \lambda^*(W \cap A_1) + \lambda^*(W \cap A_2) + \lambda^*(W \setminus (A_1 \sqcup A_2)) \\
&= \lambda^*(W \cap A_1) + \lambda^*(W \cap A_2) + \lambda^*(W \cap A_3) + \lambda^*(W \setminus (A_1 \sqcup A_2 \sqcup A_3)) \\
&= \ldots \\
&= \sum_{k=1}^{n} \lambda^*(W \cap A_k) + \lambda^*(W \setminus (A_1 \sqcup A_2 \ldots \sqcup A_n)) = \sum_{k=1}^{n} \lambda^*(W \cap A_k).
\end{aligned}$$

In the second line I used that $(W \setminus A_1) \cap A_2 = W \cap A_2$ because $A_1 \cap A_2 = \emptyset$, and $(W \setminus A_1) \setminus A_2 = W \setminus (A_1 \sqcup A_2)$. Similar identitties were used in the third line, and all the steps after that. And our simplifying assumption is used at the very end.

Of course, this is really an induction argument in disguise.

Claim: The Lebesgue measurable sets form an algebra of sets.

It follows directly from the definition that $\mathcal{M}$ is closed under complements.

We next show that it is closed under finite unions. So let $A, B \in \mathcal{M}$. For any $W \subseteq \mathbb{R}$ we find

$$\begin{aligned}
\lambda^*(W) &= \lambda^*(W \cap A) + \lambda^*(W \cap A^c) \\
&= \underbrace{\lambda^*(W \cap A \cap B) + \lambda^*(W \cap A \cap B^c) + \lambda^*(W \cap A^c \cap B)}_{\text{note: } (A\cap B)\cup(A\cap B^c)\cup(A^c\cap B) = A\cup B} + \lambda^*(W \cap A^c \cap B^c) \\
&\geq \lambda^*(W \cap (A \cup B)) + \lambda^*(W \cap (A^c \cap B^c)) \\
&= \lambda^*(W \cap (A \cup B)) + \lambda^*(W \setminus (A \cup B))
\end{aligned}$$

(using subadditivity in the third line), which shows that $A \cup B \in \mathcal{M}$. Our claim follows from this.

Claim: The Lebesgue measurable sets form a σ-algebra.

To prove this, it only remains to show that a countable union of Lebesgue measurable sets is Lebesgue measurable. So let $(E_n)_{n\in\mathbb{N}}$ be a sequence of sets, $E_n \in \mathcal{M}$, and write $E = \bigcup_{k\in\mathbb{N}} E_k$.

Furthermore, let

$$B_n = \bigcup_{k=1}^{n} E_k, \quad B_0 = \emptyset, \quad A_n = B_n \setminus B_{n-1}, \quad \text{so that } B_n = \bigsqcup_{k=1}^{n} A_k \text{ and } E = \bigsqcup_{k=1}^{\infty} A_k$$

and note that all the sets A_n, B_n are Lebesgue measurable.

Now let $W \subseteq \mathbb{R}$ be any set. I *claim*:

$$\underbrace{\lambda^*\Big(W \cap E\Big)}_{a} = \underbrace{\sum_{k=1}^{\infty} \lambda^*(W \cap A_k)}_{b} = \underbrace{\lim_{n\to\infty} \lambda^*(W \cap B_n)}_{c}.$$

The second equality ($b = c$) follows direct from the finite additivity results shown earlier, i.e.,

$$\lambda^*(W \cap B_n) = \sum_{k=1}^{n} \lambda^*(W \cap A_k),$$

and then taking the limit as $n \to \infty$. The inequality $a \le b$ is just the countable subadditivity of λ^*. The inequality $c \le a$ follows from $B_n \subseteq E$, so that $\lambda^*(W \cap B_n) \le \lambda^*(W \cap E)$. Thus the claim is proved.

Now we use the measurability of B_n:

$$\lambda^*(W) = \lambda^*(W \cap B_n) + \lambda^*(W \setminus B_n) \ge \lambda^*(W \cap B_n) + \lambda^*(W \setminus E)$$

because $B_n \subseteq E$, so $W \setminus B_n \supseteq W \setminus E$. Therefore, taking the limit and using the above claim we get

$$\lambda^*(W) \ge \lim_{n\to\infty} \lambda^*(W \cap B_n) + \lambda^*(W \setminus E) = \lambda^*(W \cap E) + \lambda^*(W \setminus E),$$

showing that $E \in \mathcal{M}$ as claimed.

Corollary. *Every open or closed set is Lebesgue measurable, and so is every Borel set.*
Proof. Any open set is a countable union of open intervals, and hence measurable. Any closed set is the complement of an open set, and hence measurable. And the set of Borel sets is the σ-algebra generated by the open subsets of $\mathbb{R}$. Since $\mathcal{M}$ is a σ-algebra containing the open subsets of $\mathbb{R}$, it contains all Borel sets.

Countable additivity of the Lebesgue measure

As a bonus from the above proof that $\mathcal{M}$ is a σ-algebra, we have

$$\lambda^*\left(W \cap \bigsqcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \lambda^*(W \cap A_n)$$

whenever $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint Lebesgue measurable sets.

We define *Lebesgue measure* λ to be the restriction of Lebesgue outer measure λ^* to the Lebesgue measurable sets:

$$\lambda(E) = \lambda^*(E), \qquad E \in \mathcal{M}.$$

The countable additivity of λ is an immediate special case of the equality above, with $W = \mathbb{R}$.