The Lebesgue integral

The point of this note is to provide a slightly different approach than the book does: I want to get to the monotone convergence theorem as quickly as possible, then use that to get all the basic properties of the integral.

I also try to simplify some proofs, where appropriate.

First, a bit of handy notation:

The *Iverson bracket* $[\cdots]$ has the value 1 if the statement inside the brackets is true, and 0 otherwise. For example, if $f:\mathbb{R}\to\mathbb{R}$ is given, then

$$[f(x)>a]=\begin{cases}1 & \text{if } f(x)>a,\\ 0 & \text{otherwise.}\end{cases}$$

The *characteristic function* (or *indicator function*) of a set A is commonly written as χ_A, as in the textbook. (One also often sees it written as $\mathbf{1}_A$ in the literature.)

It can be defined in terms of the Iverson bracket:

$$\chi_A(x)=[x\in A].$$

It is tempting to adopt the Iverson bracket for this purpose, and write $[A]$ instead of χ_A. In particular, the above definition turns into the somewhat mysterious looking

$$[A](x)=[x\in A].$$

I shall succumb to this temptation, though be warned that it is highly non-standard, and should be explained carefully whenever you use it.

As a further example, convince yourself that

$$\left[f^{-1}((a,\infty))\right](x)=[f(x)>a].$$

Integral of simple functions

A *simple function* is a function that can be written

$$\sum_{i=1}^{n} a_i\cdot[A_i]$$

where $a_i\in\mathbb{R}$ and $A_i\in\mathcal{M}$ for each $i\in\{1,\dots,n\}$.

We want to define the integral of a simple function:

$$\int_{\mathbb{R}}\sum_{i=1}^{n} a_i\cdot[A_i]\,d\lambda=\sum_{i=1}^{n} a_i\lambda(A_i).$$

We need to show that this is *well defined*, since the same simple function can be written in many different ways. (For example, the sets A_i need not be mutually disjoint.)

It is enough to show that if $\sum_{i=1}^{n} a_i\cdot[A_i]$ is the zero function then $\sum_{i=1}^{n}\lambda(A_i)=0$ (exercise: convince yourself that this is indeed enough).

To this end, define $A_i^1=A_i$ and $A_i^0=A_i^c$, and for any $b=(b_1,\dots,b_n)\in\{0,1\}^n$, define

$$B_b=\bigcap_{i=1}^{n} A_i^{b_i}.$$

Notice that the sets B_b are mutually disjoint, and their union is all of $\mathbb{R}$ (in fact $x\in B_b$ if and only if $b_i=[x\in A_i]$ for $i=1,\dots,n$). We find

$$\begin{aligned}\sum_{i=1}^{n} a_i\lambda(A_i)&=\sum_{i=1}^{n} a_i\sum_{b\in\{0,1\}^n}\lambda(B_b)\cdot[b_i=1] && A_i \text{ is a disjoint union}\dots\\ &=\sum_{b\in\{0,1\}^n}\underbrace{\sum_{i=1}^{n} a_i\cdot[b_i=1]}\,\lambda(B_b) && \text{interchange summation order}\end{aligned}$$

Now consider the term marked with a brace above, and notice that if $x\in B_b$, we get

$$\sum_{i=1}^{n} a_i\cdot[b_i=1]=\sum_{i=1}^{n} a_i\cdot[x\in A_i]=\sum_{i=1}^{n} a_i\cdot[A_i](x)=0,$$

so the marked term vanishes. If $B_b=\emptyset$, this argument does not work, but then $\lambda(B_b)=0$, so all terms of the outer sum vanish, and $\sum_{i=1}^{n} a_i\lambda(A_i)=0$. Thus we have shown that the integral is well defined for simple functions.

We shall need the obvious facts that the integral $\int_{\mathbb{R}}\varphi\,d\lambda$ is linear function of the simple function φ, and that $\int_{\mathbb{R}}\varphi\,d\lambda\ge 0$ if $\varphi\ge 0$ (because φ can be written as a sum $\sum a_k\cdot[A_i]$ with all $a_i\ge 0$).

Integral of nonnegative functions, and the monotone convergence theorem

Whenever $f\colon \mathbb{R} \to [0,\infty]$ is a measurable function, we define its integral to be

$$\int_{\mathbb{R}} f\,d\lambda = \sup\Big\{\int_{\mathbb{R}} \varphi\,d\lambda \colon 0 \le \varphi \le f \text{ and } \varphi \text{ is a simple function}\Big\}$$

In the following, we need the obvious fact that $f \le g$ implies $\int_{\mathbb{R}} f\,d\lambda \le \int_{\mathbb{R}} g\,d\lambda$.

1 Theorem (Monotone convergence theorem) *Assume that* $0 \le f_1 \le f_2 \le f_3 \le \cdots$ *are Lebesgue measurable functions, and let*

$$f(x) = \lim_{k\to\infty} f_k(x).$$

Then

$$\int_{\mathbb{R}} f\,d\lambda = \lim_{k\to\infty} \int_{\mathbb{R}} f_k\,d\lambda.$$

Proof: First, note that $\big(\int_{\mathbb{R}} f_k\,d\lambda\big)$ is a non-decreasing sequence, so it does have a limit. Also $f_k \le f$, so $\int_{\mathbb{R}} f_k\,d\lambda \le \int_{\mathbb{R}} f\,d\lambda$. Taking the limit, we conclude

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_k\,d\lambda \le \int_{\mathbb{R}} f\,d\lambda.$$

It remains to prove the opposite inequality.

Take any simple function φ with $0 \le \varphi \le f$, and any real number α with $0 < \alpha < 1$. Write

$$\varphi = \sum_{i=1}^{n} a_i \cdot [A_i], \qquad a_i > 0.$$

For any k, consider the simple function

$$\varphi_k(x) = \alpha\varphi(x) \cdot [\alpha\varphi(x) \le f_k(x)].$$

Note carefully that the bracket in the above equation is the characteristic function of a measurable set, since φ and f_k are measurable functions. Also, $0 \le \varphi_k \le f_k$ by construction, and so

$$\int_{\mathbb{R}} f_k\,d\lambda \ge \int_{\mathbb{R}} \alpha\varphi_k\,d\lambda = \sum_{i=1}^{n} \alpha a_i \cdot \lambda(\underbrace{\{x \in A_i \colon \alpha\varphi(x) \le f_k(x)\}}_{A_{ik}}).$$

The measurable sets A_{ik} form an increasing set: $A_{i1} \subseteq A_{i2} \subseteq \cdots$, and their union is all of A_i, since $\lim_{k\to\infty} f_k(x) = f(x) \ge \varphi(x) > \alpha\varphi(x)$ (note that $\varphi(x) > 0$ for $x \in A_i$). Therefore $\lim_{k\to\infty} \lambda(A_{ik}) = \lambda(A_i)$, so we get

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_k\,d\lambda \ge \sum_{i=1}^{n} \alpha a_i \cdot \lambda(A_i) = \int_{\mathbb{R}} \alpha\varphi\,d\lambda = \alpha \int_{\mathbb{R}} \varphi\,d\lambda.$$

Since $\alpha \in (0,1)$ was arbitrary, we now conclude

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_k\,d\lambda \ge \int_{\mathbb{R}} \varphi\,d\lambda,$$

and since φ was an arbitrary simple function with $0 \le \varphi \le f$, we finally have

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_k\,d\lambda \ge \int_{\mathbb{R}} f\,d\lambda,$$

and the proof is complete. ∎

Where do we go next?

From here, we pretty much follow the book. In rough outline:

Show that any nonnegative measurable function is the pointwise limit of a sequence of nonnegative simple functions, and from this derive the linearity properties of the integral.

Then extend the integral to arbitrary integrable functions, and derive the remaining convergence properties and other basic properties of the integral.