

Vitali covers

1 Definition. A *Vitali cover* of a set $E \subseteq \mathbb{R}$ is a set \mathcal{V} of closed intervals with positive length so that, for every $\delta > 0$ and every $x \in E$, there is some $I \in \mathcal{V}$ with $\lambda(I) < \delta$ and $x \in I$.

2 Lemma (Vitali covering) Given a set $E \subseteq \mathbb{R}$ with $\lambda^*(E) < \infty$, a Vitali cover \mathcal{V} of E , and some $\varepsilon > 0$, there are disjoint $I_1, \dots, I_n \in \mathcal{V}$ with $\lambda^*(E \setminus (I_1 \sqcup \dots \sqcup I_n)) < \varepsilon$.

Proof:

Start by picking an open set $O \supseteq E$ with $\lambda(O) < \infty$, and let $\mathcal{W} = \{I \in \mathcal{V} : I \subseteq O\}$. It is easy to check that \mathcal{W} is a Vitali cover of E .

We now forget about E for a while, and pick pairwise disjoint intervals I_1, I_2, \dots in \mathcal{W} . We shall do so one by one, starting with I_1 , and we shall do it *greedily*, trying to make each interval as large as possible. Now there may not be any largest size available, so we settle for getting within a constant factor:

$$\lambda(I_{k+1}) > \frac{1}{2} \sup\{\lambda(J) : J \in \mathcal{W} \text{ and } J \cap I_j = \emptyset \text{ for } j = 1, \dots, k\}. \quad (1)$$

In ordinary prose: *At each step, pick the next interval disjoint from the other intervals picked this far, so that any other interval we could have picked is less than twice as long.* It is conceivable that this process cannot go on forever. But if so, we have already covered all of E with a finite, pairwise disjoint set of intervals from \mathcal{W} , so we're done.

If $x \in E \setminus (I_1 \sqcup \dots \sqcup I_k)$, then since $I_1 \sqcup \dots \sqcup I_k$ is closed, there is some δ -neighbourhood of x that does not meet $I_1 \sqcup \dots \sqcup I_k$, and so we can continue with the selection for one more step.

If the process continues forever, then we find

$$\sum_{k=1}^{\infty} \lambda(I_k) = \lambda\left(\bigsqcup_{k=1}^{\infty} I_k\right) \leq \lambda(O) < \infty,$$

so the sum converges. Pick n large enough so that

$$\sum_{k=n+1}^{\infty} \lambda(I_k) < \varepsilon,$$

and let J_k be the interval with the same center as I_k , but five times the length.

I claim

$$E \setminus \bigsqcup_{k=1}^n I_k \subseteq \bigcup_{k=n+1}^{\infty} J_k,$$

which in turn implies

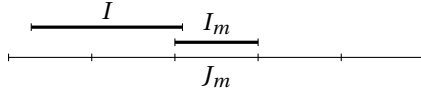
$$\lambda^*\left(E \setminus \bigsqcup_{k=1}^n I_k\right) \leq \sum_{k=n+1}^{\infty} \lambda(J_k) = 5 \sum_{k=n+1}^{\infty} \lambda(I_k) < 5\varepsilon,$$

and that is enough to finish the proof.

To prove the claim, let $x \in E \setminus (I_1 \sqcup \dots \sqcup I_n)$. Since \mathcal{W} is a Vitali cover for E , we can find some $I \in \mathcal{W}$ with $x \in I$ and $I \cap I_k = \emptyset$ for $k = 1, \dots, n$.

Let m be the smallest natural number with $I \cap I_m \neq \emptyset$.

Such an m must exist, for otherwise, I would always be one of the intervals we could have picked, and therefore $\lambda(I_{k+1}) > \frac{1}{2}\lambda(I)$ for all k . But this is impossible, since $\sum \lambda(I_k) < \infty$. There is more – when $k = m - 1$, I is still one of the J 's, so we must have $\lambda(I_m) > \frac{1}{2}\lambda(I)$.



This drawing illustrates the fact that, when $I_m \cap I \neq \emptyset$ and $\lambda(I_m) > \frac{1}{2}\lambda(I)$, then $J_m \supseteq I$. In particular, $x \in J_m$, and the proof is finally complete. ■

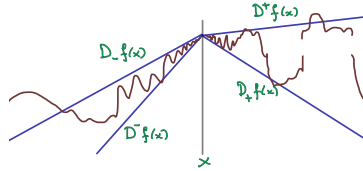
Dini derivatives

These are defined by

$$D^+ f(x) = \overline{\lim}_{y \searrow x} \frac{f(y) - f(x)}{y - x}, \quad D_+ f(x) = \underline{\lim}_{y \searrow x} \frac{f(y) - f(x)}{y - x},$$

$$D^- f(x) = \overline{\lim}_{y \nearrow x} \frac{f(y) - f(x)}{y - x}, \quad D_- f(x) = \underline{\lim}_{y \nearrow x} \frac{f(y) - f(x)}{y - x},$$

and called the *upper right*, *lower right*, *upper left*, and *lower left* Dini derivatives, respectively.



The Dini derivatives have these simple properties:

$$-\infty \leq D_{\pm} f \leq D^{\pm} f \leq \infty,$$

$$D^{\pm}(-f) = -D_{\pm} f,$$

$$D^{\pm} g = D_{\mp} f \quad \text{where } g(x) = f(-x).$$

Note that any of the Dini derivatives can take on either of the values $\pm\infty$.

Our interest in the Dini derivatives stems from the fact that they always exists, and moreover f has a left derivative x if and only if $D_- f(x) = D^- f(x)$, a right derivative if and only if $D_+ f(x) = D^+ f(x)$, and a (two-sided) derivative if and only if all four Dini derivatives are the same.

Limits to growth for a monotone function

From now on, $a < b$ are real numbers, and $f: (a, b) \rightarrow \mathbb{R}$ is an increasing function.

3 Lemma $D^+ f(x) < \infty$ and $D^- f(x) < \infty$ for λ -a.e. $x \in (a, b)$.

Proof: Let $E = \{x \in (a, b) : D^+ f(x) = \infty\}$. The idea of the proof is that f must have unbounded growth on this set, if it has positive measure. To get this to work, we will first assume that $|f| \leq M$ on (a, b) .

Pick any (large) number m . It follows from the definition that the set of intervals $[x, y]$ where

$$x \in E, \quad y \in (x, b), \quad \text{and} \quad \frac{f(y) - f(x)}{y - x} > m$$

is a Vitali cover for E . Pick any $\varepsilon > 0$, and let $[x_k, y_k]$ be pairwise disjoint intervals of this type for $k = 1, \dots, n$ with $\lambda^*(E \setminus ([x_1, y_1] \sqcup \dots \sqcup [x_n, y_n])) < \varepsilon$. It follows that $(y_1 - x_1) + \dots + (y_n - x_n) > \lambda^*(E) - \varepsilon$, and therefore

$$(f(y_1) - f(x_1)) + \dots + (f(y_n) - f(x_n)) > m(\lambda^*(E) - \varepsilon).$$

But because f is increasing, the intervals $(f(x_k), f(y_k))$ are pairwise non-overlapping in $[-M, M]$, so we must finally have $2M > m(\lambda^*(E) - \varepsilon)$. If $\lambda^*(E) > 0$, we can choose $\varepsilon < \lambda^*(E)$ and m large enough for this to be a contradiction.

If f is unbounded, we apply the above to the restriction of f to slightly smaller intervals $(a + n^{-1}, b - n^{-1})$ instead, and use the fact that a countable union of sets of measure zero still has measure zero.

We have proved that $D^+ f < \infty$ a.e. To show the same for $D^- f$, apply the first result to the function $x \mapsto -f(-x)$ on $(-b, -a)$. ■

Limits to wiggleness for a monotone function

4 Lemma $D^+ f(x) = D_+ f(x)$ and $D^- f(x) = D_- f(x)$ for λ -a.e. $x \in (a, b)$.

Proof: As in the previous lemma, the second a.e. equality follows from the first by applying it to $x \mapsto -f(-x)$. So we only prove the first one. Note that $D^+ f(x) \geq D_+ f(x)$ always, so we need to show that the set of x where $D^+ f(x) > D_+ f(x)$ has measure zero. But for any such x , we can always find two rational numbers r and s with $D^+ f(x) > s > r > D_+ f(x)$. Since the number of rational pairs (r, s) with $r < s$ is countable, we only need to show that the set

$$E = \{x \in (a, b) : D^+ f(x) > s > r > D_+ f(x)\} \quad (r, s \in \mathbb{Q} \text{ fixed})$$

has measure zero.

The proof idea is like that for the previous lemma, only done twice: Use one inequality to show that the average growth rate of f on a set of intervals nearly covering E is less than r , then use the other to find subintervals, still nearly covering E , where the average growth rate is greater than s . Then combine the two to get a contradiction, if $\lambda^*(E) > 0$.

To get started, then, we first pick some $\varepsilon > 0$, and an open set $O \supset E$ with $\lambda(O) < \lambda^*(E) + \varepsilon$. Consider the Vitali cover for E consisting of those intervals $[x, y]$ where

$$x \in E, \quad x < y, \quad y \in O, \quad \text{and} \quad \frac{f(y) - f(x)}{y - x} < r.$$

Then let $\varepsilon > 0$, and use the Vitali covering lemma to pick pairwise disjoint intervals $[x_k, y_k]$ of this type, with $\lambda^*(E \setminus ([x_1, y_1] \sqcup \cdots \sqcup [x_n, y_n])) < \varepsilon$.

It follows that

$$\sum_{k=1}^n (f(y_k) - f(x_k)) < r \sum_{k=1}^n (y_k - x_k).$$

Now let $U = (x_1, y_n) \sqcup \cdots \sqcup (x_n, y_n)$, and create yet another Vitali cover, this time of $E \cap U$, consisting of those intervals (u, v) where

$$u \in E \cap U, \quad u < v, \quad (u, v) \subseteq U, \quad \text{and} \quad \frac{f(v) - f(u)}{v - u} > s.$$

Use the Vitali lemma to almost cover $E \cap U$ with pairwise disjoint intervals $[u_j, v_j]$ of this type, with $\lambda^*(E \cap U \setminus ([u_1, v_1] \sqcup \cdots \sqcup [u_m, v_m])) < \varepsilon$. Using reasoning that should be familiar by now, we get

$$\sum_{j=1}^m (f(v_j) - f(u_j)) > s \sum_{j=1}^m (v_j - u_j).$$

But since f is increasing, the intervals $[f(u_j), f(v_j)]$ are non-overlapping. They are also each contained in some $[f(x_k), f(y_k)]$, so

$$\sum_{j=1}^m (f(v_j) - f(u_j)) \leq \sum_{k=1}^n (f(y_k) - f(x_k)).$$

From the inequalities shown we get

$$s \sum_{j=1}^m (v_j - u_j) < r \sum_{k=1}^n (y_k - x_k).$$

To estimate the right hand side, note $\bigsqcup_{k=1}^n [x_k, y_k] \subseteq O$, so

$$r \sum_{k=1}^n (y_k - x_k) \leq r\lambda(O) < r(\lambda^*(E) + \varepsilon).$$

And for the left hand side,

$$s \sum_{k=1}^m (v_j - u_j) = s\lambda\left(\bigsqcup_{j=1}^m [u_j, v_j]\right) > s(\lambda^*(E \cap U) - \varepsilon) > s(\lambda^*(E) - 2\varepsilon),$$

so now we have

$$s(\lambda^*(E) - 2\varepsilon) < r(\lambda^*(E) + \varepsilon).$$

Recall that $r < s$. So if $\lambda^*(E) > 0$, we can pick $\varepsilon > 0$ small enough that the above inequality becomes a contradiction. ■

Limited number of corners

From now on, $a < b$ are real numbers, and $f: (a, b) \rightarrow \mathbb{R}$ is an arbitrary function.

5 Definition. Call $u \in (a, b)$ a *strict local maximum point* for f if there is some $\delta > 0$ so that $f(x) < f(u)$ whenever $x \in (a, b)$ is such that $|x - u| < \delta$.

6 Lemma *The set of strict local maximum points for f is countable.*

Proof: Any two u values satisfying the definition above for some $\delta > 0$ must be at least a distance δ apart, so there is only room for a finite number of them in (a, b) . But any strict local maximum must satisfy the definition for some $\delta \in \{1/n : n \in \mathbb{N}\}$, so their total number is countable. ■

7 Definition. A *corner point* for f is a $u \in (a, b)$ so that either $D_- f(u) > D^+ f(u)$ or $D^- f(u) < D_+ f(u)$.

To understand the meaning of this definition, consult the illustration of Dini derivatives above, which shows a corner of the first kind. Turn the picture upside down to see a corner of the second kind.

8 Lemma *Any real valued function f has only a countable number of corner points.*

Proof: We only show that the set of u satisfying the first inequality is countable. The same result for the second inequality will then follow by replacing f by $-f$.

If the first inequality holds, there is some $q \in \mathbb{Q}$ with $D_- f(u) > q > D^+ f(u)$. If this holds, then x is a strict local maximum for the function $g(x) = f(x) - qx$, since then $D_- g(u) > 0 > D^+ g(u)$.

By Lemma 6 only a countable number of such u exist for any q , and since \mathbb{Q} is countable, we are done. ■

Almost everywhere differentiability

9 Theorem *Any monotone function is differentiable almost everywhere.*

Proof: We only need to consider an increasing function $f: (a, b) \rightarrow \mathbb{R}$. By Lemma 3 and 4, f has finite one-sided derivatives almost everywhere. If f has one-sided derivatives at some point, and they are different, then f has a corner point there. But Lemma 8 implies that the corner points have measure zero. ■

10 Example. Let $f: [0, 1] \rightarrow [0, 1]$ be the Cantor function, and $C \subset [0, 1]$ the (standard) Cantor set. Then f is locally constant on the open set $(0, 1) \setminus C$, so $f' = 0$ there. But $\lambda(C) = 0$, so $f' = 0$ almost everywhere. In particular,

$$\int_0^1 f' d\lambda = 0, \quad \text{and yet } f(1) - f(0) = 1.$$

This shows that we cannot expect the fundamental theorem of calculus to hold for arbitrary monotone functions – more is needed. What is happening here is that all of the growth happens within a set of measure zero.

An integral (in)equality

11 Proposition Let $f: [a, b] \rightarrow \mathbb{R}$ be an increasing function. Then f' is integrable, and

$$\int_a^b f'(x) dx \leq f(b) - f(a).$$

If there is some constant L so that f satisfies

$$f(y) \leq f(x) + L(y - x) \quad \text{whenever } a \leq x < y \leq b,$$

then equality holds instead:

$$\int_a^b f'(x) dx = f(b) - f(a).$$

By *increasing*, I mean what some prefer to call *non-decreasing*.

Proof: For simplicity, extend f by setting $f(x) = f(b)$ when $x > b$. Define $f_n: [a, b] \rightarrow \mathbb{R}$ by

$$f_n(x) = n \cdot (f(x + n^{-1}) - f(x)).$$

Then $f_n \geq 0$, and $f_n \rightarrow f'$ a.e. Therefore Fatou's lemma gives us

$$\begin{aligned} \int_a^b f'(x) dx &\leq \varliminf_{n \rightarrow \infty} \int_a^b f_n(x) dx \\ &= \varliminf_{n \rightarrow \infty} n \int_a^b (f(x + n^{-1}) - f(x)) dx \\ &= \varliminf_{n \rightarrow \infty} n \left(\int_{a+n^{-1}}^{b+n^{-1}} f(x) dx - \int_a^b f(x) dx \right) \\ &= \varliminf_{n \rightarrow \infty} \left(n \int_b^{b+n^{-1}} f(x) dx - n \int_a^{a+n^{-1}} f(x) dx \right) \\ &\leq f(b) - f(a). \end{aligned}$$

If the stated linear growth condition holds, then $f_n \leq L$, so instead of using Fatou's lemma, we can use the bounded convergence theorem (BCT), and the first inequality in the calculation above becomes equality. Also, \varliminf can be replaced by an ordinary limit. The final inequality also becomes an equality, since f is now continuous. ■