Vitali covers

1 Definition. A *Vitali cover* of a set $E\subseteq\mathbb{R}$ is a set $\mathcal{V}$ of closed intervals with positive length so that, for every $\delta>0$ and every $x\in E$, there is some $I\in\mathcal{V}$ with $\lambda(I)<\delta$ and $x\in I$.

2 Lemma (Vitali covering) *Given a set $E\subseteq\mathbb{R}$ with $\lambda^*(E)<\infty$, a Vitali cover $\mathcal{V}$ of E, and some $\varepsilon>0$, there are disjoint $I_1,\dots,I_n\in\mathcal{V}$ with $\lambda^*\big(E\setminus(I_1\sqcup\cdots\sqcup I_n)\big)<\varepsilon$.*

Proof:

Start by picking an open set $O\supseteq E$ with $\lambda(O)<\infty$, and let $\mathcal{W}=\{I\in\mathcal{V}\colon I\subseteq O\}$. It is easy to check that $\mathcal{W}$ is a Vitali cover of E.

We now forget about E for a while, and pick pairwise disjoint intervals I_1, I_2, … in $\mathcal{W}$. We shall do so one by one, starting with I_1, and we shall do it *greedily*, trying to make each interval as large as possible. Now there may not be any largest size available, so we settle for getting within a constant factor:

$$\lambda(I_{k+1})>\tfrac{1}{2}\sup\big\{\lambda(J)\colon J\in\mathcal{W}\text{ and }J\cap I_j=\emptyset\text{ for }j=1,\dots,k\big\}.\tag{1}$$

In ordinary prose: *At each step, pick the next interval disjoint from the other intervals picked this far, so that any other interval we could have picked is less than twice as long.* It is conceivable that this process cannot go on forever. But if so, we have already covered all of E with a finite, pairwise disjoint set of intervals from $\mathcal{W}$, so we're done.

If $x\in E\setminus(I_1\sqcup\cdots\sqcup I_k)$, then since $I_1\sqcup\cdots\sqcup I_k$ is closed, there is some δ-neighbourhood of x that does not meet $I_1\sqcup\cdots\sqcup I_k$, and so we can continue with the selection for one more step.

If the process continues forever, then we find

$$\sum_{k=1}^{\infty}\lambda(I_k)=\lambda\Big(\bigsqcup_{k=1}^{\infty}I_k\Big)\le\lambda(O)<\infty,$$

so the sum converges. Pick n large enough so that

$$\sum_{k=n+1}^{\infty}\lambda(I_k)<\varepsilon,$$

and let J_k be the interval with the same center as I_k, but five times the length.

I claim

$$E\setminus\bigsqcup_{k=1}^{n}I_k\subseteq\bigcup_{k=n+1}^{\infty}J_k,$$

which in turn implies

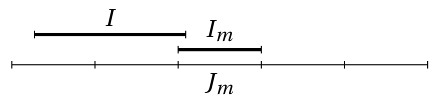
$$\lambda^*\Big(E\setminus\bigsqcup_{k=1}^{n}I_k\Big)\le\sum_{k=n+1}^{\infty}\lambda(J_k)=5\sum_{k=n+1}^{\infty}\lambda(I_k)<5\varepsilon,$$

and that is enough to finish the proof.

To prove the claim, let $x\in E\setminus(I_1\sqcup\cdots\sqcup I_n)$. Since $\mathcal{W}$ is a Vitali cover for E, we can find some $I\in\mathcal{W}$ with $x\in I$ and $I\cap I_k=\emptyset$ for $k=1,\dots,n$.

Let m be the smallest natural number with $I\cap I_m\neq\emptyset$.

Such an m must exist, for otherwise, I would always be one of the intervals we *could* have picked, and therefore $\lambda(I_{k+1})>\frac{1}{2}\lambda(I)$ for all k. But this is impossible, since $\sum\lambda(I_k)<\infty$. There is more – when $k=m-1$, I is still one of the J's, so we must have $\lambda(I_m)>\frac{1}{2}\lambda(I)$.



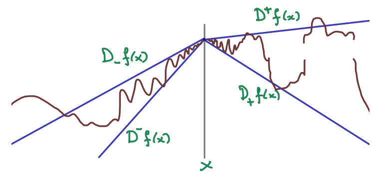
This drawing illustrates the fact that, when $I_m\cap I\neq\emptyset$ and $\lambda(I_m)>\frac{1}{2}\lambda(I)$, then $J_m\supseteq I$. In particular, $x\in J_m$, and the proof is finally complete. ∎

Dini derivatives

These are defined by

$$D^+f(x)=\overline{\lim_{y\searrow x}}\frac{f(y)-f(x)}{y-x},\qquad D_+f(x)=\underline{\lim_{y\searrow x}}\frac{f(y)-f(x)}{y-x},$$
$$D^-f(x)=\overline{\lim_{y\nearrow x}}\frac{f(y)-f(x)}{y-x},\qquad D_-f(x)=\underline{\lim_{y\nearrow x}}\frac{f(y)-f(x)}{y-x},$$

and called the *upper right*, *lower right*, *upper left*, andn *lower left* Dini derivatives, respectively.



The Dini derivatives have these simple properties:

$$\begin{aligned}-\infty\le D_\pm f&\le D^\pm f\le\infty,\\ D^\pm(-f)&=-D_\pm f,\\ D^\pm g&=D_\mp f\qquad\text{where }g(x)=f(-x).\end{aligned}$$

Note that any of the Dini derivatives can take on either of the values $\pm\infty$.

Our interest in the Dini derivatives stems from the fact that they always exists, and moreover f has a left derivative x if and only if $D_-f(x)=D^-f(x)$, a right derivative if and only if $D_+f(x)=D^+f(x)$, and a (two-sided) derivative if and only if all four Dini derivatives are the same.

Limits to growth for a monotone function

From now on, $a < b$ are real numbers, and $f\colon (a,b) \to \mathbb{R}$ is an increasing function.

3 Lemma *$D^+f(x) < \infty$ and $D^-f(x) < \infty$ for λ-a.e. $x \in (a,b)$.*

Proof: Let $E = \{x \in (a,b)\colon D^+f(x) = \infty\}$. The idea of the proof is that f must have unbounded growth on this set, if it has positive measure. To get this to work, we will first assume that $|f| \le M$ on (a,b).

Pick any (large) number m. It follows from the definition that the set of intervals $[x,y]$ where

$$x \in E, \quad y \in (x,b), \quad \text{and } \frac{f(y)-f(x)}{y-x} > m$$

is a Vitali cover for E. Pick any $\varepsilon > 0$, and let $[x_k, y_k]$ be pairwise disjoint intervals of this type for $k = 1,\ldots,n$ with $\lambda^*(E \setminus ([x_1,y_1] \sqcup \cdots \sqcup [x_n,y_n])) < \varepsilon$. It follows that $(y_1 - x_1) + \cdots + (y_n - x_n) > \lambda^*(E) - \varepsilon$, and therefore

$$\big(f(y_1) - f(x_1)\big) + \cdots + \big(f(y_n) - f(x_n)\big) > m(\lambda^*(E) - \varepsilon).$$

But because f is increasing, the intervals $(f(x_k), f(y_k))$ are pairwise non-overlapping in $[-M,M]$, so we must finally have $2M > m(\lambda^*(E)-\varepsilon)$. If $\lambda^*(E) > 0$, we can choose $\varepsilon < \lambda^*(E)$ and m large enough for this to be a contradiction.

If f is unbounded, we apply the above to the restriction of f to slightly smaller intervals $(a + n^{-1}, b - n^{-1})$ instead, and use the fact that a countable union of sets of measure zero still has measure zero.

We have proved that $D^+f < \infty$ a.e. To show the same for D^-f, apply the first result to the function $x \mapsto -f(-x)$ on $(-b,-a)$. ∎

Limits to wiggliness for a monotone function

4 Lemma *$D^+f(x) = D_+f(x)$ and $D^-f(x) = D_-f(x)$ for λ-a.e. $x \in (a,b)$.*

Proof: As in the previous lemma, the second a.e. equality follows from the first by applying it to $x \mapsto -f(-x)$. So we only prove the first one. Note that $D^+f(x) \ge D_+f(x)$ always, so we need to show that the set of x where $D^+f(x) > D_+f(x)$ has measure zero. But for any such x, we can always find two rational numbers r and s with $D^+f(x) > s > r > D_+f(x)$. Since the number of rational pairs (r,s) with $r < s$ is countable, we only need to show that the set

$$E = \{x \in (a,b)\colon D^+f(x) > s > r > D_+f(x)\} \qquad (r,s \in \mathbb{Q} \text{ fixed})$$

has measure zero.

The proof idea is like that for the previous lemma, only done twice: Use one inequality to show that the average growth rate of f on a set of intervals nearly covering E is less than r, then use the other to find subintervals, still nearly covering E, where the average growth rate is greater than s. Then combine the two to get a contradiction, if $\lambda^*(E) > 0$.

To get started, then, we first pick some $\varepsilon > 0$, and an open set $O \supset E$ with $\lambda(O) < \lambda^*(E) + \varepsilon$. Consider the Vitali cover for E consisting of those intervals $[x,y]$ where

$$x \in E, \quad x < y, \quad y \in O, \quad \text{and } \frac{f(y)-f(x)}{y-x} < r.$$

Then let $\varepsilon > 0$, and use the Vitali covering lemma to pick pairwise disjoint intervals $[x_k, y_k]$ of this type, with with $\lambda^*(E \setminus ([x_1,y_1] \sqcup \cdots \sqcup [x_n,y_n])) < \varepsilon$.

It follows that

$$\sum_{k=1}^{n} \big(f(y_k) - f(x_k)\big) < r \sum_{k=1}^{n} (y_k - x_k).$$

Now let $U = (x_1, y_n) \sqcup \cdots \sqcup (x_n, y_n)$, and create yet another Vitali cover, this time of $E \cap U$, consisting of those intervals (u,v) where

$$u \in E \cap U, \quad u < v, \quad (u,v) \subseteq U, \quad \text{and } \frac{f(v)-f(u)}{v-u} > s.$$

Use the Vitali lemma to almost cover $E \cap U$ with pairwise disjoint intervals $[u_j, v_j]$ of this type, with $\lambda^*(E \cap U \setminus ([u_1,v_1] \sqcup \cdots \sqcup [u_m,v_m])) < \varepsilon$. Using reasoning that should be familiar by now, we get

$$\sum_{j=1}^{n} \big(f(v_j) - f(u_j)\big) > s \sum_{j=1}^{n} (v_j - u_j).$$

But since f is increasing, the intervals $[f(u_j), f(v_j)]$ are non-overlapping. They are also each contained in some $[f(x_k), f(y_k)]$, so

$$\sum_{j=1}^{n} \big(f(v_j) - f(u_j)\big) \le \sum_{k=1}^{n} \big(f(y_k) - f(x_k)\big).$$

From the inequalities shown we get

$$s\sum_{j=1}^{m}(v_j-u_j)<r\sum_{k=1}^{n}(y_k-x_k).$$

To estimate the right hand side, note $\bigsqcup_{k=1}^{n}[x_k,y_k]\subseteq O$, so

$$r\sum_{k=1}^{n}(y_k-x_k)\le r\lambda(O)<r\big(\lambda^*(E)+\varepsilon\big).$$

And for the left hand side,

$$s\sum_{k=1}^{m}(v_j-u_j)=s\lambda\Big(\bigsqcup_{j=1}^{m}[u_j,v_j]\Big)>s\big(\lambda^*(E\cap U)-\varepsilon\big)>s\big(\lambda^*(E)-2\varepsilon\big),$$

so now we have

$$s\big(\lambda^*(E)-2\varepsilon\big)<r\big(\lambda^*(E)+\varepsilon\big).$$

Recall that $r<s$. So if $\lambda^*(E)>0$, we can pick $\varepsilon>0$ small enough that the above inequality becomes a contradiction. ∎

Limited number of corners

From now on, $a<b$ are real numbers, and $f\colon (a,b)\to\mathbb{R}$ is an arbitrary function.

5 Definition. Call $u\in(a,b)$ a *strict local maximum point* for f if there is some $\delta>0$ so that $f(x)<f(u)$ whenever $x\in(a,b)$ is such that $|x-u|<\delta$.

6 Lemma *The set of strict local maximum points for f is countable.*

Proof: Any two u values satisfying the definition above for some $\delta>0$ must be at least a distance δ apart, so there is only room for a finite number of them in (a,b). But any strict local maximum must satisfy the definition for some $\delta\in\{1/n\colon n\in\mathbb{N}\}$, so their total number is countable. ∎

7 Definition. A *corner point* for f is a $u\in(a,b)$ so that either $D_-f(u)>D^+f(u)$ or $D^-f(u)<D_+f(u)$.

To understand the meaning of this definition, consult the illustration of Dini derivatives above, which shows a corner of the first kind. Turn the picture upside down to see a corner of the second kind.

8 Lemma *Any real valued function f has only a countable number of corner points.*

Proof: We only show that the set of u satisfying the first inequality is countable. The same result for the second inequality will then follow by replacing f by $-f$.

If the first inequality holds, there is some $q\in\mathbb{Q}$ with $D_-f(u)>q>D^+f(u)$. If this holds, then x is a strict local maximum for the function $g(x)=f(x)-qx$, since then $D_-g(u)>0>D^+g(u)$.

By Lemma 6 only a countable number of such u exist for any q, and since $\mathbb{Q}$ is countable, we are done. ∎

9 Theorem *Any monotone function is differentiable almost everywhere.*

Proof: We only need to consider an increasing function $f\colon (a,b) \to \mathbb{R}$. By Lemma 3 and 4, f has finite one-sided derivatives almost everywhere. If f has one-sided derivatives at some point, and they are different, then f has a corner point there. But Lemma 8 implies that the corner points have measure zero. ∎

10 Example. Let $f\colon [0,1] \to [0,1]$ be the Cantor function, and $C \subset [0,1]$ the (standard) Cantor set. Then f is locally constant on the open set $(0,1) \setminus C$, so $f' = 0$ there. But $\lambda(C) = 0$, so $f' = 0$ almost everywhere. In particular,

$$\int_0^1 f'\,d\lambda = 0, \quad \text{and yet } f(1) - f(0) = 1.$$

This shows that we cannot expect the fundamental theorem of calculus to hold for arbitrary monotone functions – more is needed. What is happening here is that all of the growth happens within a set of measure zero.

11 Proposition *Let $f\colon [a,b] \to \mathbb{R}$ be an increasing function. Then f' is integrable, and*

$$\int_a^b f'(x)\,dx \le f(b) - f(a).$$

If there is some constant L so that f satisfies

$$f(y) \le f(x) + L(y-x) \qquad \text{whenever } a \le x < y \le b,$$

then equality holds instead:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

By *increasing*, I mean what some prefer to call *non-decreasing.*

Proof: For simplicity, extend f by setting $f(x) = f(b)$ when $x > b$. Define $f_n\colon [a,b] \to \mathbb{R}$ by

$$f_n(x) = n \cdot \bigl(f(x+n^{-1}) - f(x)\bigr).$$

Then $f_n \ge 0$, and $f_n \to f'$ a.e. Therefore Fatou's lemma gives us

$$\begin{aligned}
\int_a^b f'(x)\,dx &\le \varliminf_{n\to\infty} \int_a^b f_n(x)\,dx \\
&= \varliminf_{n\to\infty} n \int_a^b \bigl(f(x+n^{-1}) - f(x)\bigr)\,dx \\
&= \varliminf_{n\to\infty} n\Bigl(\int_{a+n^{-1}}^{b+n^{-1}} f(x)\,dx - \int_a^b f(x)\,dx\Bigr) \\
&= \varliminf_{n\to\infty} \Bigl(n\int_b^{b+n^{-1}} f(x)\,dx - n\int_a^{a+n^{-1}} f(x)\,dx\Bigr) \\
&\le f(b) - f(a).
\end{aligned}$$

If the stated linear growth condition holds, then $f_n \le L$, so instead of using Fatou's lemma, we can use the bounded convergence theorem (BCT), and the first inequality in the calculation above becomes equality. Also, $\varliminf$ can be replace by an ordinary limit. The final inequality also becomes an equality, since f is now continuous. ∎