Compactness

Compact subsets of the real line: A subset $K \subseteq \mathbb{R}$ is called *compact* if, whenever $\mathcal{O}$ is a set of open subsets of $\mathbb{R}$ with $\bigcup \mathcal{O} \supseteq K$, there is a finite subset $\mathcal{Q} \subset \mathcal{O}$ with $\bigcup \mathcal{Q} \supseteq K$.

$\mathcal{O}$ is called an *open cover* of K, and $\mathcal{Q}$ is called a *finite subcover*.

To be more precise, we might wish to specify that $\mathcal{Q}$ is a subcover of K, or of $\mathcal{O}$. But the language gets really awkward if we have to specify both K and $\mathcal{O}$. Perhaps then it is better to say that $\mathcal{Q} \subseteq \mathcal{O}$ is a (sub)cover of K. We might also say that $\mathcal{Q} \subseteq \mathcal{O}$ covers K.

Characterization of compact sets: *A subset of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.*

Proof. An unbounded subset of $\mathbb{R}$ has an open cover consisting of all bounded, open intervals. This has no finite subcover, since the union of a finite set of bounded intervals is bounded.

Similarly, if K is not closed then it has a boundary point $x \notin K$. But then the collection of sets $\mathbb{R} \setminus [x-\varepsilon, x+\varepsilon]$ with $\varepsilon > 0$ is an open cover with no finite subcover.

We have shown that a set which is either unbounded or not closed is not compact. It remains to prove that a set $K \subset \mathbb{R}$ which is both bounded and closed is compact.

First, we show that a closed and bounded interval $[a, b]$ is compact.

Let $\mathcal{O}$ be an open cover of $[a, b]$.

Define

$$s = \sup C \quad \text{where} \quad C = \{c \in [a,b] : \mathcal{O} \text{ has a finite subcover for } [a,c]\}.$$

It is clear that $a \in C$, since $\mathcal{O}$ is a cover for $[a, b]$. So C is not empty, and $s \in [a, b]$.

Pick some $U \in \mathcal{O}$ with $s \in U$, and let $\varepsilon > 0$ so that $[s-\varepsilon, s+\varepsilon] \subseteq U$.

Pick some $c \in C$ with $c > s - \varepsilon$ (it exists by the definition of s). Then there is a finite subcover $\mathcal{Q}$ of $\mathcal{O}$ for $[a, c]$. But then $\mathcal{Q} \cup \{U\}$ is a finite subcover of $\mathcal{O}$ for $[a, c']$, where $c' = \min(s+\varepsilon, b)$. Thus $c' \in C$. If $s < b$ then this contradicts the definition of s, therefore $s = b$, and $b = c' \in C$, in other words there is a finite subcover for $[a, b]$.

Finally, let $K \subset \mathbb{R}$ be closed and bounded, and let $\mathcal{O}$ be an open cover of K. Then $U = \mathbb{R} \setminus K$ is open, and $\mathcal{O} \cup \{U\}$ is an open cover of $[a, b]$, where $K \subseteq [a, b]$ (which can be so arranged because K is bounded). Since $[a, b]$ is compact, it has a finite subcover $\mathcal{Q}$. Now $\mathcal{Q} \setminus \{U\} \subseteq \mathcal{O}$ is a finite cover of K, and the proof is complete.

Sequential compactness: *A subset K of the real line is compact if, and only if, every sequence inside K has a cluster point which lies in K.*

The proof below is not the simplest for subsets of the real line, but it has the advantage of being easily generalizable to $\mathbb{R}^n$, or indeed any metric space.

Proof. Assume that K is compact. Let (x_n) be a sequence in K. If $w \in K$ is not a cluster point of (x_n), there is some $\varepsilon > 0$ and $N \in \mathbb{N}$ so that $|w - x_n| \geq \varepsilon$ whenever $n \geq N$. Assume that (x_n) has no cluster point in K. Then, by compactness, K is covered by a finite set of intervals, say $(w_i - \varepsilon_i, w_i + \varepsilon_i)$ for $i \in \{1, \ldots, m\}$, where $|w_i - x_n| \geq \varepsilon$ whenever $n \geq N_i$. But then, if $n \geq \max(N_1, \ldots, N_m)$, $|w_i - x_n| \geq \varepsilon$ for all $i \in \{1, \ldots, m\}$. But this is a contradiction since the intervals $(w_i - \varepsilon_i, w_i + \varepsilon_i)$ cover K.

Conversely, assume that every sequence in K has a cluster point in K. Also, assume that $\mathscr{O}$ is an open cover of K containing no finite subcover.

First, we claim that for every $\varepsilon > 0$ there is some $x \in K$ so that $(x - \varepsilon, x + \varepsilon) \nsubseteq O$ for all $O \in \mathcal{O}$. For otherwise, pick sequences (x_n) in K and $(O_n) \in \mathcal{O}$ so that $x_n \notin O_1 \cup \cdots \cup O_{n-1}$ and $(x_n - \varepsilon, x_n + \varepsilon) \subseteq O_n$. These conditions imply $|x_m - x_n| \geq \varepsilon$ for all $m < n$, and such a sequence can have no cluster point. This contradiction proves the claim.

Now let (ε_n) be a sequence of positive numbers converging to zero (for example, $\varepsilon_n = 1/n$), and for each n pick $x_n \in K$ with $(x_n - \varepsilon_n, x_n + \varepsilon_n) \nsubseteq O$ for all $O \in \mathcal{O}$.

The sequence (x_n) must have a cluster point w in K. Pick $O \in \mathscr{O}$ with $w \in O$. Since O is open, there is some $\varepsilon > 0$ with $(w - \varepsilon, w + \varepsilon) \subseteq O$. Since w is a cluster point of (x_n), and $\varepsilon_n \to 0$, we can find some n with $|x_n - w| < \varepsilon/2$ and $\varepsilon_n < \varepsilon/2$. But then $(x_n - \varepsilon_n, x_n + \varepsilon_n) \subset (w - \varepsilon, w + \varepsilon) \subseteq O$, and this contradicts the choice of x_n and ε_n.