Compact subsets of the real line: A subset $K \subseteq \mathbb{R}$ is called *compact* if, whenever $\mathcal{O}$ is a set of open subsets of $\mathbb{R}$ with $\bigcup \mathcal{O} \supseteq K$, there is a finite subset $\mathcal{Q} \subset \mathcal{O}$ with $\bigcup \mathcal{Q} \supseteq K$.

$\mathcal{O}$ is called an *open cover* of $K$, and $\mathcal{Q}$ is called a *finite subcover*.

To be more precise, we might wish to specify that $\mathcal{Q}$ is a subcover of $K$, or of $\mathcal{O}$. But the language gets really awkward if we have to specify both $K$ and $\mathcal{O}$. Perhaps then it is better to say that $\mathcal{Q} \subseteq \mathcal{O}$ is a (sub)cover of $K$. We might also say that $\mathcal{Q} \subseteq \mathcal{O}$ covers $K$. 
**Characterization of compact sets:** A subset of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.

*Proof.* An unbounded subset of $\mathbb{R}$ has an open cover consisting of all bounded, open intervals. This has no finite subcover, since the union of a finite set of bounded intervals is bounded.

Similarly, if $K$ is not closed then it has a boundary point $x \notin K$. But then the collection of sets $\mathbb{R} \setminus [x - \epsilon, x + \epsilon]$ with $\epsilon > 0$ is an open cover with no finite subcover.

We have shown that a set which is either unbounded or not closed is not compact. It remains to prove that a set $K \subset \mathbb{R}$ which is both bounded and closed is compact.

First, we show that a closed and bounded interval $[a, b]$ is compact.

Let $\emptyset$ be an open cover of $[a, b]$.

Define

$$s = \sup C \quad \text{where} \quad C = \{c \in [a, b]: \emptyset \text{ has a finite subcover for } [a, c]\}.$$  

It is clear that $a \in C$, since $\emptyset$ is a cover for $[a, b]$. So $C$ is not empty, and $s \in [a, b]$.

Pick some $U \in \emptyset$ with $s \in U$, and let $\epsilon > 0$ so that $[s - \epsilon, s + \epsilon] \subseteq U$.

Pick some $c \in C$ with $c > s - \epsilon$ (it exists by the definition of $s$). Then there is a finite subcover $Q$ of $\emptyset$ for $[a, c]$. But then $Q \cup \{U\}$ is a finite subcover of $\emptyset$ for $[a, c']$, where $c' = \min(s + \epsilon, b)$. Thus $c' \in C$. If $s < b$ then this contradicts the definition of $s$, therefore $s = b$, and $b = c' \in C$, in other words there is a finite subcover for $[a, b]$.

Finally, let $K \subset \mathbb{R}$ be closed and bounded, and let $\emptyset$ be an open cover of $K$. Then $U = \mathbb{R} \setminus K$ is open, and $\emptyset \cup \{U\}$ is an open cover of $[a, b]$, where $K \subseteq [a, b]$ (which can be so arranged because $K$ is bounded). Since $[a, b]$ is compact, it has a finite subcover $Q$. Now $Q \setminus \{U\} \subseteq \emptyset$ is a finite cover of $K$, and the proof is complete.
**Sequential compactness:** A subset $K$ of the real line is compact if, and only if, every sequence inside $K$ has a cluster point which lies in $K$.

The proof below is not the simplest for subsets of the real line, but it has the advantage of being easily generalizable to $\mathbb{R}^n$, or indeed any metric space.

**Proof.** Assume that $K$ is compact. Let $(x_n)$ be a sequence in $K$. If $w \in K$ is not a cluster point of $(x_n)$, there is some $\varepsilon > 0$ and $N \in \mathbb{N}$ so that $|w - x_n| \geq \varepsilon$ whenever $n \geq N$. Assume that $(x_n)$ has no cluster point in $K$. Then, by compactness, $K$ is covered by a finite set of intervals, say $(w_i - \varepsilon_i, w_i + \varepsilon_i)$ for $i \in \{1, \ldots, m\}$, where $|w_i - x_n| \geq \varepsilon$ whenever $n \geq N_i$. But then, if $n \geq \max(N_1, \ldots, N_m)$, $|w_i - x_n| \geq \varepsilon$ for all $i \in \{1, \ldots, m\}$. But this is a contradiction since the intervals $(w_i - \varepsilon_i, w_i + \varepsilon_i)$ cover $K$.

Conversely, assume that every sequence in $K$ has a cluster point in $K$. Also, assume that $\emptyset$ is an open cover of $K$ containing no finite subcover.

First, we claim that for every $\varepsilon > 0$ there is some $x \in K$ so that $(x - \varepsilon, x + \varepsilon) \not\subseteq O$ for all $O \in \emptyset$. For otherwise, pick sequences $(x_n)$ in $K$ and $(O_n) \in \emptyset$ so that $x_n \notin O_1 \cup \cdots \cup O_{n-1}$ and $(x_n - \varepsilon, x_n + \varepsilon) \subseteq O_n$. These conditions imply $|x_m - x_n| \geq \varepsilon$ for all $m < n$, and such a sequence can have no cluster point. This contradiction proves the claim.

Now let $(\varepsilon_n)$ be a sequence of positive numbers converging to zero (for example, $\varepsilon_n = 1/n$), and for each $n$ pick $x_n \in K$ with $(x_n - \varepsilon_n, x_n + \varepsilon_n) \not\subseteq O$ for all $O \in \emptyset$.

The sequence $(x_n)$ must have a cluster point $w$ in $K$. Pick $O \in \emptyset$ with $w \in O$. Since $O$ is open, there is some $\varepsilon > 0$ with $(w - \varepsilon, w + \varepsilon) \subseteq O$. Since $w$ is a cluster point of $(x_n)$, and $\varepsilon_n \to 0$, we can find some $n$ with $|x_n - w| < \varepsilon/2$ and $\varepsilon_n < \varepsilon/2$. But then $(x_n - \varepsilon_n, x_n + \varepsilon_n) \subseteq (w - \varepsilon, w + \varepsilon) \subseteq O$, and this contradicts the choice of $x_n$ and $\varepsilon_n$. 