Coin tossing space

Think of a *coin toss* as a random choice from the two element set $\{0,1\}$. Thus the set $\{0,1\}^n$ represents the set of possible outcomes of n coin tosses, and $\Omega := \{0,1\}^{\mathbb{N}}$, consisting of all sequences $(t_n)_{n\in\mathbb{N}}$, represents the set of possible outcomes of tossing a coin infinitely many times.

If the coin is unbiased and the tosses are independent, each of the 2^n outcomes in $\{0,1\}^n$ is equally probable. So $\{0,1\}^n$ is equipped with a probability measure μ_n, which is 2^{-n} times counting measure on $\{0,1\}^n$. (All subsets of $\{0,1\}^n$ are measurable.)

Compatible probability measures

Assume that $m < n$. Tossing a coin n times, and then ignoring all but the first m of them, is represented by a *projection map* $\pi_{mn}\colon \{0,1\}^n \to \{0,1\}^m$:

$$\pi_{mn}(t_1,\dots,t_n) = (t_1,\dots,t_m).$$

An event depending on $(t_1,\dots,t_m)$ is a set $E \subseteq \{0,1\}^m$. It has probability $\mu_m(E)$. The "same" event considered as a subset of $\{0,1\}^n$ is $\pi_{mn}^{-1}(E)$. It does in fact have the same probability, in $\{0,1\}^n$, as E does, in $\{0,1\}^m$:

$$\mu_n\bigl(\pi_{mn}^{-1}(E)\bigr) = \mu_m(E), \qquad E \subseteq \{0,1\}^m. \tag{1}$$

Think of this as a compatibility result: All the different probability measures μ_n give the same result for those events they can measure.

(*Exercise*: Show this. *Hint*: Show that for each $s \in E$ there are 2^{n-m} distinct sequences $t \in \pi_{mn}^{-1}(E)$ with $\pi_{mn}(t) = s$.)

Lots and lots of projections

All these projection maps π_{nm} are compatible in the sense that

$$\pi_{km} \circ \pi_{mn} = \pi_{kn}, \qquad k < m < n.$$

It is helpful to picture the finite coin tossing spaces as an infinite sequence:

$$\{0,1\} \leftarrow \{0,1\}^2 \leftarrow \{0,1\}^3 \leftarrow \cdots \leftarrow \{0,1\}^n \leftarrow \{0,1\}^{n+1} \leftarrow \cdots$$

where the arrows shown are π_{12}, π_{23}, ..., $\pi_{n,n+1}$. At the far right we can put the infinite coin tossing space Ω. (Category theorists call that the *projective limit* of the above diagram, but never mind that.) There is also a projection map $\pi_n \colon \Omega \to \{0,1\}^n$, given by $\pi_n(t_1, t_2, \ldots) = (t_1, \ldots, t_n)$. Clearly, the compatibility of projections extends to this one:

$$\pi_{mn} \circ \pi_n = \pi_m, \qquad m < n. \tag{2}$$

Finitely determined events and their probabilities

Some events in Ω depend only on a finite number of coin tosses. (By this I mean that to determine whether $t \in \Omega$ belongs to some event E, you only need to look at $t_1, \dots, t_n$ for some n, and that n is independent of t.) If $E \subseteq \{0,1\}^n$ then

$$\pi_n^{-1}(E) = \{t \in \Omega \colon (t_1, \dots, t_n) \in E\}$$

is such an event, and I expect you can convince yourself that every such "finitely determined" event has this form. (If not, consider that a *definition*.) Define

$$\mathcal{A}_n = \{\pi_n^{-1}(E) \colon E \subseteq \{0,1\}^n\}, \qquad n \in \mathbb{N}.$$

Then $\mathcal{A}_n$ is a σ-algebra of subsets of Ω for each $n \in \mathbb{N}$, and moreover $\mathcal{A}_m \subset \mathcal{A}_n$ when $m < n$ (for the equality $\pi_{mn} \circ \pi_n = \pi_m$ implies $\pi_m^{-1}(E) = \pi_n^{-1}(\pi_{nm}^{-1}(E))$ when $E \subseteq \{0,1\}^m$).

Let

$$\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n.$$

It is not hard to show that, since $(\mathcal{A}_n)$ is an increasing sequence of algebras, the union $\mathcal{C}$ is also an algebra of sets (do it!). We can define a function ι on $\mathcal{C}$ by

$$\iota\big(\pi_n^{-1}(E)\big) = \mu_n(E), \qquad E \subseteq \{0,1\}^n.$$

(*Exercise*: Show that this is well defined. *Hint*: Use (1) and (2).)

Extending the measure

The function ι represents the probability of *any* finitely determined event. But what about other events? How can we discuss the probability that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} t_k = \frac{1}{2},$$

for example? The set of $t \in \Omega$ satisfying this equality is clearly not finitely determined.

What is needed is to extend ι to a measure. It is defined on the algebra $\mathcal{C}$, which is *not* a σ-algebra. But $\mathcal{C}$ is a semialgebra, so in order to show that we can extend ι to a measure, we need to show that $\iota(\emptyset) = 0$, that ι is finitely additive, and the appropriate subadditivity condition.

For the first property, $\iota(\emptyset) = \mu_n(\pi_n^{-1}(\emptyset)) = \mu_n(\emptyset) = 0$.

For the second, if $F_1,\ldots,F_m \in \mathcal{C}$ are pairwise disjoint then there is some n so that $F_k \in \mathcal{A}_n$ for $k = 1,\ldots,m$. Write $F_k = \pi_n^{-1}(E_k)$. Then the sets E_k must be pairwise disjoint, and

$$\iota\Big(\bigsqcup_{k=1}^{m} F_k\Big) = \mu_n\Big(\bigsqcup_{k=1}^{m} E_k\Big) = \sum_{k=1}^{m}\mu_n(E_k) = \sum_{k=1}^{m}\iota(F_k).$$

Finally, assume that $F, F_k \in \mathcal{C}$ with $F \subseteq \bigcup_{k\in\mathbb{N}} F_k$. I *claim* that a finite number of the sets F_k will cover F, i.e., $F \subseteq F_1 \cup\cdots\cup F_n$ for some n. If this is true, then

$$\iota(F) \le \iota\Big(\bigcup_{k=1}^{n} F_k\Big) \le \sum_{k=1}^{n}\iota(F_k) \le \sum_{k=1}^{\infty}\iota(F_k),$$

which is the subadditivity condition we need to show.

To prove the claim, assume instead it is false. That is, we assume that each set A_n, defined by $A_n = F \setminus (F_1 \cup\cdots\cup F_n)$, is nonempty. Note that $A_n \in \mathcal{A}$, and $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. We now define a sequence (u_k) in $\{0,1\}$ by induction.

First, since $\Omega = \pi_1^{-1}\{(0)\} \cup \pi_1^{-1}\{(1)\}$, we find $A_n = \big(A_n \cap \pi_1^{-1}\{(0)\}\big) \cup \big(A_n \cap \pi_1^{-1}\{(1)\}\big)$, so we can select $u_1 \in \{0,1\}$ so that $A_n \cap \pi_1^{-1}\{(u_1)\} \neq \emptyset$ for infinitely many n, and therefore for all $n \in \mathbb{N}$, since (A_n) is a decreasing sequence of sets.

Next, we get $A_n \cap \pi_1^{-1}\{(u_1)\} = \big(A_n \cap \pi_2^{-1}\{(u_1,0)\}\big) \cup \big(A_n \cap \pi_1^{-1}\{(u_1,1)\}\big)$, so we can select $u_2 \in \{0,1\}$ so that $A_n \cap \pi_1^{-1}\{(u_1,u_2)\} \neq \emptyset$ for infinitely many n, and therefore for all $n \in \mathbb{N}$.

Continuing this procedure, we end up with a sequence $u = (u_k)_{k\in\mathbb{N}}$ with the property that, for each k, $A_n \cap \pi_k^{-1}\{(u_1,\ldots,u_k)\} \neq \emptyset$ for all $n \in \mathbb{N}$.

I claim that $u \in \cap_{n\in\mathbb{N}} A_n$. To see this, merely note that A_n (for any n) is finititely determined, i.e., there is some $k \in \mathbb{N}$ so that $A_n = \pi_k^{-1}E$ with $E \subseteq \{0,1\}^k$. But since $A_n \cap \pi_k^{-1}\{(u_1,\ldots,u_k)\} \neq \emptyset$, it follows that $(u_1,\ldots,u_k) \in E$, and therefore $u \in A_n$. We have proved that $\cap_{n\in\mathbb{N}} A_n \neq \emptyset$. This contradicts the assumption that $F \subseteq \cup_{k\in\mathbb{N}} F_k$, and the claim is therefore proved.

Probability on the coin tossing space

From the general theory, we now have a measure μ on a σ-algebra $\mathcal{A}$ extending $\mathcal{C}$, and hence all the algebras $\mathcal{A}_n$. Moreover, by construction $\mu\big(\pi_n^{-1}(E)\big) = \mu_n(E)$ for all $E \in \{0,1\}^n$, so our probability measure certainly captures our intuition about the distribution of n coin tosses.

The nth toss is represented by a function $T_n \colon \Omega \to \{0,1\}$, defined by $T_n(t) = t_n$. T_n is a *stochastic variable*, i.e., a measurable function on Ω. It is not difficult to show that the tosses T_n are stochastically independent: If $G_1, \dots, G_n \subseteq \mathbb{R}$ are Borel sets, then

$$\mu\Big(\bigcap_{k=1}^{n} T_k^{-1}(G_k)\Big) = \prod_{k=1}^{n} \mu\big(T_k^{-1}(G_k)\big).$$

(Note that $T_k^{-1}(G_k)$ is one of four sets: $\emptyset$, $\{t \in \Omega\colon\ t_k = 0\}$, $\{t \in \Omega\colon\ t_k = 1\}$, or Ω.)