The Carathéodory construction of measures

This note is essentially a copy of an earlier note on the same construction of the Lebesgue measure on the real line, just replacing the outer Lebesgue measure λ^* by a general outer measure μ^*.

Historical note: The construction outlined here is due to Constantin Carathéodory (Κωνσταντίνος Καραθεοδωρή).

Outer measure

An *outer measure* on a set Ω is a map $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ satisfying these properties:

- $\mu^*(\emptyset) = 0$
- (Monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$
- (Subadditivity) For any sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}$,

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$$

In the remainder of this note, the set Ω and an outer measure μ^* on Ω will be fixed.

Carathéodory’s criterion and measurability

We say that a set $E \subseteq \Omega$ is μ^* *-measurable* if

$$\mu^*(W) = \mu^*(W \cap E) + \mu^*(W \setminus E) \qquad \text{for all } W \subseteq \Omega.$$

The criterion above is known as *Carathéodory’s criterion.*

In the rest of this note I write just “measurable”, since μ^* is fixed.

It is useful to observe that the inequality “$\leq$” holds by subaddtivity, so to prove that a set is measurable, we only need to prove

$$\mu^*(W) \geq \mu^*(W \cap E) + \mu^*(W \setminus E) \qquad \text{for all } W \subseteq \Omega.$$

We write $\mathcal{A}$ for the set of measurable sets.

A finite additivity result

For any pairwise disjoint sets $A_1, A_2, \ldots, A_n$ with $A_1, A_2, \ldots, A_{n-1} \in \mathcal{A}$ and any $W \subseteq \Omega$, we have

$$\mu^*\left(W \cap \bigsqcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu^*(W \cap A_k).$$

A word on notation: I use the notation $\bigsqcup$ to indicate a *disjoint union,* i.e., a union of pairwise disjoint sets. We use this symbol, and its binary variant $\sqcup$, for the union *only* when the sets involved are known (or assumed) to be pairwise disjoint. So $A \sqcup B = A \cup B$ if $A \cap B = \emptyset$, but the notation $A \sqcup B$ should be considered to be meaningless otherwise.

Proof: We prove it first for $n = 2$: If A, B are pairwise disjoint and $A \in \mathcal{A}$, then

$$\mu^*(W \cap (A \sqcup B)) = \mu^*(W \cap A) + \mu^*(W \cap B),$$

which follows directly from the measurability of A because $(W \cap (A \sqcup B)) \cap A = W \cap A$ and $(W \cap (A \sqcup B)) \setminus A = W \cap B$.

In the general case, using the above with $A = A_1$ and $B = \bigsqcup_{k=2}^{n} A_k$ yields

$$\mu^*\left(W \cap \bigsqcup_{k=1}^{n} A_k\right) = \mu^*(W \cap A_1) + \mu^*\left(W \cap \bigsqcup_{k=2}^{n} A_k\right),$$

and applying this inductively to the last term yields the desired result.

***Claim*: The measurable sets form an algebra of sets.**

It follows directly from the definition that $\mathcal{A}$ is closed under complements.

We next show that it is closed under finite unions. So let $A, B \in \mathcal{A}$. For any $W \subseteq \Omega$ we find

$$\begin{aligned}
\lambda^*(W) &= \lambda^*(W \cap A) + \lambda^*(W \cap A^c) \\
&= \overbrace{\underbrace{\lambda^*(W \cap A \cap B) + \lambda^*(W \cap A \cap B^c) + \lambda^*(W \cap A^c \cap B)}_{\text{note: } (A\cap B)\cup(A\cap B^c)\cup(A^c\cap B) = A\cup B}}^{} + \lambda^*(W \cap A^c \cap B^c) \\
&\geq \lambda^*(W \cap (A \cup B)) + \lambda^*(W \cap (A^c \cap B^c)) \\
&= \lambda^*(W \cap (A \cup B)) + \lambda^*(W \setminus (A \cup B))
\end{aligned}$$

(using subadditivity in the third line), which shows that $A \cup B \in \mathcal{A}$. Our claim follows from this.

A countable additivity result

For any sequence $(A_k)_{k\in\mathbb{N}}$ *of pairwise disjoint measurable sets and any* $W \subseteq \Omega$*, we have*

$$\mu^*\left(W \cap \bigsqcup_{k\in\mathbb{N}} A_k\right) = \sum_{k\in\mathbb{N}} \mu^*(W \cap A_k). \tag{1}$$

Proof: We use our finite additivity result, noting that we do not yet know, nor do we need to know, that $\bigsqcup_{k=n+1}^{\infty} A_k$ is measurable:

$$\begin{aligned}
\mu^*\left(W \cap \bigsqcup_{k\in\mathbb{N}} A_k\right) &= \mu^*\left(\left(W \cap \bigsqcup_{k=1}^{n} A_k\right) \sqcup \left(W \cap \bigsqcup_{k=n+1}^{\infty} A_k\right)\right) \\
&= \sum_{k=1}^{n} \mu^*(W \cap A_k) + \mu^*\left(\left(W \cap \bigsqcup_{k=n+1}^{\infty} A_k\right)\right) \\
&\geq \sum_{k=1}^{n} \mu^*(W \cap A_k),
\end{aligned}$$

and letting $n \to \infty$ we conclude that

$$\mu^*\left(W \cap \bigsqcup_{k\in\mathbb{N}} A_k\right) \geq \sum_{k=1}^{\infty} \mu^*(W \cap A_k).$$

Since the opposite inequality holds by countable subadditivity, the proof is complete.

***Claim*: The measurable sets form a σ-algebra.**

To prove this, it only remains to show that a countable union of measurable sets is measurable.

So let $(E_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets, and write

$$E = \bigcup_{k\in\mathbb{N}} E_k.$$

Furthermore, let

$$B_n = \bigcup_{k=1}^{n} E_k, \quad B_0 = \emptyset, \quad A_n = B_n \setminus B_{n-1}, \quad \text{so that } B_n = \bigsqcup_{k=1}^{n} A_k \text{ and } E = \bigsqcup_{k=1}^{\infty} A_k$$

and note that all the sets A_n, B_n are measurable.

Now let $W \subseteq \Omega$ be any set. Then

$$\begin{aligned} \mu^*(W) &= \mu^*(W \cap B_n) + \mu^*(W \setminus B_n) && B_n \text{ is measurable} \\ &\geq \mu^*(W \cap B_n) + \mu^*(W \setminus E) && B_n \subseteq E, \text{ so } W \setminus B_n \supseteq W \setminus E \end{aligned}$$

Now let $n \to \infty$:

$$\begin{aligned} \mu^*(W) &\geq \lim_{n\to\infty} \mu^*(W \cap B_n) + \mu^*(W \setminus E) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu^*(W \cap A_n) + \mu^*(W \setminus E) && \text{by finite additivity} \\ &= \sum_{k=1}^{\infty} \mu^*(W \cap A_n) + \mu^*(W \setminus E) \\ &= \mu^*(W \cap E) + \mu^*(W \setminus E) && \text{by countable additivity,} \end{aligned}$$

showing that $E \in \mathcal{A}$ as claimed.

We *really* only needed countable *sub*additivity in the last line to obtain the needed inequality.

The measure induced by the outer measure

Recall equation (1). With $W = \Omega$, it becomes

$$\mu^*\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu^*(A_n)$$

whenever $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint measurable sets.

We can therefore μ to be the restriction of outer measure μ^* to the measurable sets:

$$\mu(E) = \mu^*(E), \qquad E \in \mathcal{A},$$

and conclude that μ is indeed a measure on $\mathcal{A}$.