Introduction

The customary definition of a sequence is as an indexed family $(x_n)_{n\in\mathbb{N}}$. This unnecessary insistence that the index set be all of $\mathbb{N}$ (the set of natural numbers) causes some difficulties with subsequences: The conventional definition of a subsequence is a family $(y_k)_{k\in\mathbb{N}}$ where $y_k = x_{n_k}$ and $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence of natural numbers. This seems only a minor difficulty at first, but it is compounded when we encounter a sequence of sequences, each of which is a subsequence of the one before it, and need to pick a common subsequence for the whole. The resulting indexing nightmare has surely caused headaches for many students.

Sequences

1 Definition. A *sequence* is a function defined on an infinite set I of natural numbers. The set I will be called the *index set* of the sequence. We use the customary index notation for sequences, writing $(x_n)_{n\in I}$ for a sequence defined on the index set I.

So we can define sequences on the set of all natural numbers, or the even numbers, the powers of 2, or the prime numbers.

The usual definition of convergence applies: A sequence $(x_n)_{n\in I}$ of real numbers is said to *converge* to a limit a if, for each $\varepsilon > 0$, there is some natural number N so that $n \ge N$ and $n \in I$ imply $|x_n - a| < \varepsilon$. We will also write that $x_n \to a$ as $I \ni n \to \infty$. (We may read the latter formula as "$n \to \infty$ through I".)

The notation introduced above is admittedly rather ugly, so we might prefer to shorten it to $x_n \to a$ $(n \in I)$, thus leaving the somewhat redundant "$n \to \infty$" to the imagination of the reader.

Many other concepts transfer without difficulty from the conventional theory of sequences, in which the index set is all of $\mathbb{N}$. So we can talk of cluster points, limits superior and inferior, the Cauchy property of real sequences, and so forth.

But the notion of *subsequences* becomes easier with the present approach than the conventional one, which is why we do it this way.

Subsequences

2 Definition. A *proper subsequence* of a sequence $(x_n)_{n \in I}$ is a sequence of the form $(x_n)_{n \in J}$ where $J \subseteq I$. In other words, we obtain a proper subsequence by restricting the original sequence to a smaller index set.

Since we can throw out any finite number of indices without changing the convergence properties of any sequence, a more relaxed definition is actually more useful:

3 Definition. A (possibly improper) *subsequence* of a sequence $(x_n)_{n \in I}$ is a sequence of the form $(x_n)_{n \in J}$ where all but a finite number of the members of J belong to I – or more concisely, where $J \setminus I$ is a finite set.

It is understood that x_n is the same in both sequences, whenever $n \in I \cap J$.

The following lemma is utterly trivial, but utterly essential as well.

4 Lemma *Any subsequence of a convergent sequence is itself convergent, with the same limit.*

The “diagonal” lemma

I should perhaps apologize to the reader for the name given to the following lemma. You may well ask, “diagonal? *What* diagonal?”. Fair enough – the name stems from the usual proof of the lemma in the more conventional treatment of sequences.

5 Lemma (Diagonal lemma) *Let $(x_n)_{n\in I_m}$ be a sequence for each $m\in\mathbb{N}$, such that $(x_n)_{n\in I_{m+1}}$ is a subsequence of $(x_n)_{n\in I_m}$ for $m\in\mathbb{N}$. Then there exists a common subsequence $(x_n)_{n\in J}$ of all the given sequences.*

Proof: Let $J=\{n_1,n_2,\dots\}$ where n_m is the smallest member of

$$I_1\cap I_2\cap\cdots\cap I_m\cap\{m,m+1,\dots\}.$$

Since $n_k\in I_m$ for $k\geq m$, it follows that $(x_n)_{n\in J}$ is a subsequence of $(x_n)_{n\in I_m}$ for each m. ∎

We took $n_m\geq m$ in the proof only in order to ensure that J is infinite. It could be that some n_m for different m coincide, but that does not create any difficulty.

We cannot expect to find a common *proper* subsequence for all the given sequences. This fact is the reason we needed the more general definition of (non-proper) subsequence given initially.

Example

6 Proposition *If a sequence of real numbers has a cluster point, then some subsequence converges to that cluster point.*

Proof: Let $(x_n)_{n\in I}$ be a sequence with a cluster point c. We shall find a subsequence converging to c. Define

$$I_k = \{n \in I \colon |x_n - c| < 1/k\}.$$

It is an easy exercise to show that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$, and each I_k is infinite because c is a cluster point. Let $(x_n)_{n\in J}$ be a common subsequence of all the subsequences $(x_n)_{n\in I_k}$.

Given some $\varepsilon > 0$, pick k with $1/k \le \varepsilon$. By the definition of I_k, $|x_n - c| < \varepsilon$ whenever $n \in I_k$. And since $J \setminus I_k$ is finite, this holds for all sufficiently large members of J. ∎

Another example

Our second example is the proof of a countable version of Tychonov's theorem, that the product of compact spaces (using $[0,1]$ as the canonical example) is compact.

7 Proposition *Let $(x_n)_{n\in I}$ be a sequence of functions $x_n\colon \mathbb{N} \to [0,1]$. Then there exists a subsequence $(x_n)_{n\in J}$ so that $(x_n(m))_{n\in J}$ converges in $[0,1]$ for all $m \in \mathbb{N}$.*

Proof: By the compactness of $[0,1]$, there exists an index set $I_1 \subseteq I$ so that $x_n(1)$ converges as $I_1 \ni n \to \infty$. Next, there is some $I_2 \subseteq I_1$ so that $x_n(2)$ converges as $I_2 \ni n \to \infty$, and so forth. In summary, we have a descending family of index sets

$$I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

so that $x_n(m) \to y(m)$, say, as $I_m \ni n \to \infty$.

By Lemma 5, there is a common subsequence $(x_n)_{n\in J}$ of all the subsequences $(x_n)_{n\in I_m}$.

Now fix some $m \in \mathbb{N}$. It follows from the above that $(x_n(m))_{n\in J}$ is a subsequence of $(x_n(m))_{n\in I_m}$, and so it is convergent by Lemma 4. ∎

Relation to the standard notion of sequences

As mentioned initially, the standard notion of sequences insists that the index set be $\mathbb{N}$ rather than a subset.

Given a sequence $(x_n)_{n\in J}$ where $J \subseteq \mathbb{N}$ is infinite, we can always enumerate the members of J in increasing order:

$$J = \{n_1, n_2, n_3, \ldots\} \qquad \text{where } n_1 < n_2 < n_3 < \cdots.$$

Moreover, there is only one possible increasing enumeration of J, so we get a one-to-one correspondence between infinite subsets of $\mathbb{N}$ on one hand and strictly increasing sequences (indexed by $\mathbb{N}$) of natural numbers on the other.

We can now *relabel* the given sequence $(x_n)_{n\in J}$ as $(x_{n_k})_{k\in\mathbb{N}}$. It lists the same items in the same order as the original, but now the index set is $\mathbb{N}$. We might say the sequence is on *standard form.*

Next, if we start out with a sequence $(x_n)_{n\in\mathbb{N}}$ on standard form and then take a subsequence $(x_n)_{n\in J}$, and then apply the above relabeling procedure to the subsequence, we get a standard sequence $(x_{n_k})_{k\in\mathbb{N}}$ with $n_1 < n_2 < \cdots$. This is the conventional definition of a subsequence.

You can easily imaging the notational difficulties we get into when working with an infinitely descending sequence of subsequences. Basically, it goes like this: Start with a sequence $(x_n)_{n\in\mathbb{N}}$. A subsequence might be written (y_{1k}) where $y_{1k} = x_{n_{1k}}$, $n_{11} < n_{12} < n_{13} < \cdots$. And a subsequence of *that* might be written (y_{2k}) where $y_{2k} = x_{n_{1n_{2k}}}$, $n_{21} < n_{22} < n_{23} < \cdots$. At the next step we are looking at $y_{3k} = x_{n_{1n_{2n_{3k}}}}$ – and the notation becomes hopelessly involved. The y_{jk} notation works better, though, where j records the level of subsequence and k is the index at that level. The diagonal lemma earns its name from the consideration of the "diagonal" sequence $(y_{kk})_{k\in\mathbb{N}}$.