Egorov’s theorem

1. Theorem (Egorov) Let \((\Omega, \mathcal{A}, \mu)\) be a measure space, let \(E \subseteq \Omega\) with \(E \in \mathcal{A}\) and \(\mu(E) < \infty\), and let \(f_n, f : E \to \mathbb{R}\) be measurable functions so that \(f_n \to f\) \(\mu\)-a.e. on \(E\). Let \(\varepsilon > 0\) Then there is some \(A \subseteq E\) with \(A \in \mathcal{A}\) and \(\mu(A) < \varepsilon\) so that \(f_n \to f\) uniformly on \(E \setminus A\).

Egorov’s theorem is also known as one of Littlewood’s principles:

Pointwise convergence is almost uniform.

– but note that this principle holds only on sets of finite measure.

Proof: For each \(\eta > 0\), let

\[
A_\eta = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in E : |f_n(x) - f(x)| \geq \eta\}.
\]

Whenever \(f_n(x) \to f(x), x \in A_\eta\). Thus \(\mu(A_\eta) = 0\). Since \(E \supseteq A_{\eta,N} \setminus A_\eta\) and \(\mu(E) < \infty\), \(\mu(A_{\eta,N}) \to 0\) when \(N \to \infty\).

For each \(k \in \mathbb{N}\), pick \(N_k \in \mathbb{N}\) with \(\mu(A_{1/k,N_k}) < 2^{-k}\varepsilon\), and let \(A = \bigcup_{k=1}^{\infty} A_{1/k,N_k}\). Clearly, \(\mu(A) < \varepsilon\).

I claim that \(f_n \to f\) uniformly on \(E \setminus A\). To see this, let \(\eta > 0\) and pick \(k > 1/\eta\). If \(x \in E \setminus A\) and \(n \geq N_k\), then \(x \in E \setminus A_{1/k,N_k}\), so \(|f_n(x) - f(x)| < 1/k < \eta\).
Lusin’s theorem – another one of Littlewood’s principles

Littlewood stated is thus:

Every measurable function is almost continuous.

There are several ways to state this as a precise theorem, all (or mostly) bearing Lusin’s name. We shall state and prove two of these.

But first, some lemmas.
A short series of lemmas

The following lemma is also one of Littlewood’s principles:

A Lebesgue measurable set of finite measure is almost a finite union of intervals.

**2 Lemma** Given a Lebesgue measurable set $E \subset \mathbb{R}$ with $\lambda(E) < \infty$ and $\varepsilon > 0$, there exists a finite union $K$ of intervals with $\lambda(E \triangle K) < \varepsilon$.

**Proof:** By the definition of outer Lebesgue measure, there is a sequence of intervals $I_k$ with $E \subset \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \lambda(I_k) < \lambda(E) + \varepsilon$. Since the series converges, we can find $n$ with $\sum_{k=n+1}^{\infty} \lambda(I_k) < \varepsilon$. Let $K = \bigcup_{k=1}^{n} I_k$. Then $E \setminus K \subseteq \bigcup_{k=n+1}^{\infty} I_k$, so $\lambda(E \setminus K) < \varepsilon$. Also, $K \setminus E \subseteq \bigcup_{k=1}^{\infty} I_k \setminus E$, so $\lambda(K \setminus E) \leq \lambda(\bigcup_{k=1}^{n+1} I_k) - \lambda(E) < \varepsilon$. Thus $\lambda(E \triangle K) = \lambda(E \setminus K) + \lambda(K \setminus E) < 2\varepsilon$.

Recall that a *simple function* is a measurable function which takes only a finite number of values, and which vanishes outside a set of finite measure. In particular, it is a finite linear combination of characteristic functions of measurable sets of finite measure.

By a *step function* we shall mean a finite linear combination of characteristic functions of bounded intervals.

**3 Lemma** Given a simple function $s$ and $\varepsilon > 0$, there is a step function $S$ so that $\lambda(\{x \in \mathbb{R} : S(x) \neq s(x)\}) < \varepsilon$.

**Proof:** This is really a corollary to the previous lemma. Let $s = \sum_{k=1}^{n} \chi_{A_k}$, let $K_k$ be a finite union of intervals with $\lambda(A_k \triangle K_k) < \varepsilon/n$, and let $S = \sum_{k=1}^{n} \chi_{K_k}$. Then $S(x) = s(x)$ except for $x \in \bigcup_{k=1}^{n} (A_k \triangle K_k)$, and this set has measure $< \varepsilon$.

**4 Lemma** If $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue measurable, there is a sequence of step functions converging to $f$ almost everywhere.

**Proof:** Take a sequence $(s_n)$ of simple functions so that $s_n \to f$ pointwise. For each $n$, let $S_n$ be a step function so that $S_n = s_n$ except on a set $D_n$, where $\lambda(D_n) < 2^{-n}$. Let

$$D = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} D_n,$$

and note that $\lambda(D) = 0$, because $\lambda(\bigcup_{n=N}^{\infty} D_n) \leq \sum_{n=N}^{\infty} \lambda(D_n) < \sum_{n=N}^{\infty} 2^{-n} = 2^{1-N}$.

If $x \in \mathbb{R} \setminus D$ then for some $N$, $S_n(x) = s_n(x)$ for all $n \geq N$, so that $S_n(x) \to f(x)$ because $s_n(x) \to f(x)$. 

Lusin's theorem, take one

5 Theorem (Lusin) If \( f : \mathbb{R} \rightarrow \mathbb{C} \) is Lebesgue measurable and \( \varepsilon > 0 \), there is some Lebesgue measurable set \( E \subseteq \mathbb{R} \) with \( \lambda(E) < \varepsilon \) so that \( f\mid_{\mathbb{R}\setminus E} \) is continuous.

Note that this theorem does not claim that \( f \) is continuous at every \( x \in \mathbb{R} \setminus E \). It is the restriction of \( f \) that is continuous. To illustrate the difference, consider \( f = \chi_\mathbb{Q} \), which is nowhere continuous. However, its restriction to \( \mathbb{R} \setminus \mathbb{Q} \) is continuous: This restriction is constantly zero.

Proof: Take a sequence \((S_n)\) of step functions converging a.e. to \( f \). For each integer \( N \), Egorov’s theorem implies the existence of a measurable set \( A_N \subseteq (N, N+1) \) with \( \lambda(A_N) < 2^{-|N|} \varepsilon \) so that \( S_n \rightarrow f \) uniformly on \((N, N+1) \setminus A_N \). Let \( A = \bigcup_{N \in \mathbb{Z}} A_n \). Then \( \lambda(A) < 3\varepsilon \). Further, let \( D \) be the set of points where some \( S_n \) is discontinuous. Since each \( S_n \) is discontinuous at only a finite number of points, \( D \) is countable. Let \( E = \mathbb{Z} \cup D \cup A \). Then \( \lambda(E) = \lambda(A) < 3\varepsilon \). Each \( S_n \) is continuous on \( E \), and the convergence to \( f \) is uniform on \((N, N+1) \setminus E \), so the restriction of \( f \) to \((N, N+1) \setminus E \) is continuous. Therefore, so is the restriction of \( f \) to \( \mathbb{R} \setminus E \).

That last bit is subtle, so let us do it in more detail: If \( x \in \mathbb{R} \setminus E \), then \( x \notin \mathbb{Z} \), so \( x \in (N, N+1) \) for some \( N \in \mathbb{Z} \). Given \( \varepsilon > 0 \) there is some \( \delta > 0 \) so that, whenever \( y \in (N, N+1) \setminus E \) and \( |y - z| < \delta \), then \( |f(y) - f(x)| < \varepsilon \). Here we can replace \( \delta \) by a smaller number, so that the \( \delta \)-neighbourhood of \( x \) is contained in \((N, N+1) \). Then, if \( y \in \mathbb{R} \setminus E \) and \( |y - x| < \delta \), we also have \( y \in (N, N+1) \setminus E \), so that \( |f(y) - f(x)| < \varepsilon \).
Lusin’s theorem, take two

Theorem 5 is good as far as it goes, but we can improve on it:

6 Theorem (Lusin)  If $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue measurable and $\varepsilon > 0$, there is some Lebesgue measurable set $G \subset \mathbb{R}$ with $\lambda(G) < \varepsilon$ and a continuous function $g : \mathbb{R} \to \mathbb{C}$ so that $f(x) = g(x)$ for all $x \in \mathbb{R} \setminus G$.

Proof:  Begin by finding a set $E$ as in Theorem 5, so that $\lambda(E) < \varepsilon$, and $f\big|_{\mathbb{R} \setminus E}$ is continuous.

Pick an open set $G \supseteq E$ with $\lambda(G) < 2\varepsilon$. Then also $f\big|_{\mathbb{R} \setminus G}$ is continuous. We shall expand this function to a function $g : \mathbb{R} \to \mathbb{C}$. So we begin by defining $g(x) = f(x)$ when $x \in \mathbb{R} \setminus G$.

The set $G$, being open, is a pairwise disjoint union of open intervals – the components of $G$. Say,

$$ G = \bigsqcup_{k=1}^{\infty} (a_k, b_k). $$

Expand $g$ by making it linear on $[a_k, b_k]$:

$$ g(x) = \frac{(b_k - x)f(a_k) + (x - a_k)f(b_k)}{b_k - a_k} \quad \text{for } x \in (a_k, b_k). $$

Then $g$ is indeed continuous on $\mathbb{R}$ (exercise!), and $g = f$ on $\mathbb{R} \setminus G$. $\blacksquare$
Approximation in $L^1$

We don’t need the full power of Lusin’s theorem to prove the following approximation result. But it does seem like a nice application.

7 Theorem Let $f \in L^1(\lambda)$ and $\varepsilon > 0$. Then there is a continuous function $g \in L^1(\lambda)$ with

$$\int_{\mathbb{R}} |f - g| d\lambda < \varepsilon.$$ 

Proof: We prove this only for $f \colon \mathbb{R} \to \mathbb{R}$. If $f$ is complex valued, we can apply the real result to the real and imaginary parts of $f$ separately.

Let $B_n = \{x \in \mathbb{R} : |f(x)| \geq n\}$. Since $\chi_{B_n} f \to 0$ as $n \to \infty$, and $|\chi_{B_n} f| \leq |f|$, we can use the dominated convergence theorem (DCT) to conclude that $\int_{B_n} |f| d\lambda \to 0$ when $n \to \infty$. Pick some $n$ so that $\int_{B_n} |f| d\lambda < \varepsilon$.

Now define $f_n$ by

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq n, \\ n & \text{if } f(x) > n, \\ -n & \text{if } f(x) < -n. \end{cases}$$

Pick a continuous function $g \colon \mathbb{R} \to \mathbb{R}$ so that $f(x) = g(x)$ for all $x$ except for $x$ in a set $A$ with $\lambda(A) < \varepsilon/n$. We may assume that $|g| \leq n$ everywhere, for otherwise, redefine it by setting $g(x) = n$ where before $g(x) > n$ and $g(x) = -n$ where before $g(x) < -n$. Now

$$\int_{\mathbb{R}} |f - g| d\lambda \leq \int_{\mathbb{R}} |f - f_n| d\lambda + \int_{\mathbb{R}} |f_n - g| d\lambda$$

$$= \int_{B_n} |f - f_n| d\lambda + \int_{A} |f_n - g| d\lambda$$

$$\leq \int_{B_n} |f| d\lambda + 2n \lambda(A)$$

$$< \varepsilon + 2\varepsilon = 3\varepsilon.$$

Finally, $g \in L^1(\lambda)$ because the triangle inequality yields

$$\int_{\mathbb{R}} |g| d\lambda \leq \int_{\mathbb{R}} |g - f| d\lambda + \int_{\mathbb{R}} |f| d\lambda < \infty.$$ 

The proof is now complete. □