The Carathéodory construction of measures

This note is essentially a copy of an earlier note on the same construction of the Lebesgue measure on the real line, just replacing the outer Lebesgue measure λ^* by a general outer measure μ^*.

Historical note: The construction outlined here is due to Constantin Carathéodory (Κωνσταντίνος Καραθεοδωρή).

Outer measure

An *outer measure* on a set Ω is a map $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ satisfying these properties:

- $\mu^*(\emptyset) = 0$
- (Monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$
- (Subadditivity) For any sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}$,

$$\mu^*\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$$

In the remainder of this note, the set Ω and an outer measure μ^* on Ω will be fixed.

Carathéodory's criterion and measurability

We say that a set $E \subseteq \Omega$ is μ^**-measurable* if

$$\mu^*(W) = \mu^*(W \cap E) + \mu^*(W \setminus E) \qquad \text{for all } W \subseteq \Omega.$$

The criterion above is known as *Carathéodory's criterion.*

In the rest of this note I write just "measurable", since μ^* is fixed.

It is useful to observe that the inequality "$\leq$" holds by subaddtivity, so to prove that a set is measurable, we only need to prove

$$\mu^*(W) \geq \mu^*(W \cap E) + \mu^*(W \setminus E) \qquad \text{for all } W \subseteq \Omega.$$

We write $\mathcal{A}$ for the set of measurable sets.

A finite additivity result

For any pairwise disjoint measurable sets $A_1, A_2, \ldots, A_n$ *and any* $W \subseteq \Omega$*, we have*

$$\mu^*\Big(W \cap \bigsqcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \mu^*(W \cap A_k).$$

A word on notation: I use the notation $\bigsqcup$ to indicate a *disjoint union*, i.e., a union of pairwise disjoint sets. We use this symbol, and its binary variant $\sqcup$, for the union *only* when the sets involved are known (or assumed) to be pairwise disjoint. So $A \sqcup B = A \cup B$ if $A \cap B = \emptyset$, but the notation $A \sqcup B$ should be considered to be meaningless otherwise.

Proof: Neither side of the equality changes if we replace W by $W \cap \bigsqcup_{k=1}^{n} A_k$, so we might as well assume that $W \subseteq \bigsqcup_{k=1}^{n} A_k$. Now we find

$$\begin{aligned}
\mu^*(W) &= \mu^*(W \cap A_1) + \mu^*(W \setminus A_1) \\
&= \mu^*(W \cap A_1) + \mu^*(W \cap A_2) + \mu^*(W \setminus (A_1 \sqcup A_2)) \\
&= \mu^*(W \cap A_1) + \mu^*(W \cap A_2) + \mu^*(W \cap A_3) + \mu^*(W \setminus (A_1 \sqcup A_2 \sqcup A_3)) \\
&= \ldots \\
&= \sum_{k=1}^{n} \mu^*(W \cap A_k) + \mu^*(W \setminus (A_1 \sqcup A_2 \ldots \sqcup A_n)) = \sum_{k=1}^{n} \mu^*(W \cap A_k).
\end{aligned}$$

In the second line I used that $(W \setminus A_1) \cap A_2 = W \cap A_2$ because $A_1 \cap A_2 = \emptyset$, and $(W \setminus A_1) \setminus A_2 = W \setminus (A_1 \sqcup A_2)$. Similar identities were used in the third line, and all the steps after that. And our simplifying assumption is used at the very end.

Of course, this is really an induction argument in disguise.

***Claim*: The measurable sets form an algebra of sets.**

It follows directly from the definition that $\mathcal{A}$ is closed under complements.

We next show that it is closed under finite unions. So let $A, B \in \mathcal{A}$. For any $W \subseteq \Omega$ we find

$$\begin{aligned}
\mu^*(W) &= \mu^*(W \cap A) + \mu^*(W \cap A^c) \\
&= \underbrace{\mu^*(W \cap A \cap B) + \mu^*(W \cap A \cap B^c) + \mu^*(W \cap A^c \cap B)}_{\text{note: } (A\cap B)\cup(A\cap B^c)\cup(A^c\cap B)=A\cup B} + \mu^*(W \cap A^c \cap B^c) \\
&\geq \mu^*(W \cap (A \cup B)) + \mu^*(W \cap (A^c \cap B^c)) \\
&= \mu^*(W \cap (A \cup B)) + \mu^*(W \setminus (A \cup B))
\end{aligned}$$

(using subadditivity in the third line), which shows that $A \cup B \in \mathcal{A}$. Our claim follows from this.

Claim: The measurable sets form a σ-algebra.

To prove this, it only remains to show that a countable union of measurable sets is measurable. So let $(E_n)_{n\in\mathbb{N}}$ be a sequence of sets, $E_n \in \mathcal{A}$, and write $E = \bigcup_{k\in\mathbb{N}} E_k$.

Furthermore, let

$$B_n = \bigcup_{k=1}^{n} E_k, \quad B_0 = \emptyset, \quad A_n = B_n \setminus B_{n-1}, \quad \text{so that } B_n = \bigsqcup_{k=1}^{n} A_k \text{ and } E = \bigsqcup_{k=1}^{\infty} A_k$$

and note that all the sets A_n, B_n are measurable.

Now let $W \subseteq \Omega$ be any set. I *claim*:

$$\underbrace{\mu^*\Big(W \cap E\Big)}_{a} = \underbrace{\sum_{k=1}^{\infty} \mu^*(W \cap A_k)}_{b} = \underbrace{\lim_{n\to\infty} \mu^*(W \cap B_n)}_{c}.$$

The second equality ($b = c$) follows direct from the finite additivity results shown earlier, i.e.,

$$\mu^*(W \cap B_n) = \sum_{k=1}^{n} \mu^*(W \cap A_k),$$

and then taking the limit as $n \to \infty$. The inequality $a \le b$ is just the countable subadditivity of μ^*. The inequality $c \le a$ follows from $B_n \subseteq E$, so that $\mu^*(W \cap B_n) \le \mu^*(W \cap E)$. Thus the claim is proved.

Now we use the measurability of B_n:

$$\mu^*(W) = \mu^*(W \cap B_n) + \mu^*(W \setminus B_n) \ge \mu^*(W \cap B_n) + \mu^*(W \setminus E)$$

because $B_n \subseteq E$, so $W \setminus B_n \supseteq W \setminus E$. Therefore, taking the limit and using the above claim we get

$$\mu^*(W) \ge \lim_{n\to\infty} \mu^*(W \cap B_n) + \mu^*(W \setminus E) = \mu^*(W \cap E) + \mu^*(W \setminus E),$$

showing that $E \in \mathcal{A}$ as claimed.

The measure induced by the outer measure

As a bonus from the above proof that $\mathcal{A}$ is a σ-algebra, we have

$$\mu^*\Big(W \cap \bigsqcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \mu^*(W \cap A_n)$$

whenever $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint measurable sets.

We now *define a measure* μ to be the restriction of outer measure μ^* to the measurable sets:

$$\mu(E) = \mu^*(E), \qquad E \in \mathcal{A}.$$

The countable additivity of μ is an immediate special case of the equality above, with $W = \Omega$.