TMA4225 Foundations of Analysis 2011-10-28

Solution

Problem 1

a. Any intersection of closed sets is closed. The set in question is

$$\bigcap_{n\in\mathbb{N}} F_n \cap [-100, 100]$$

which is closed and bounded. .... True

b. Consider $F_n = [1/n, 2]$ .... False

Problem 2

a. Consider the set of singleton sets: $\{\{x\}: x \in \mathbb{R}\}$ .... False

b. Each nonempty member contains rational points which are not in any of the others .... True

c. Let $K \subset \mathbb{R}$ be compact. Then $\mathbb{R} \setminus K$ is open, and hence a countable union of open intervals, say

$$\mathbb{R} \setminus K = \bigcup_{n\in\mathbb{N}} I_n.$$

Two of these intervals will be $(-\infty, a)$ and (b, ∞), where $a = \inf K$ and $b = \sup K$. We may assume they are J_0 and J_1. Then

$$F = \bigcap_{n\geq 2} \big([a, b] \setminus J_n\big)$$

where each set in the intersection is the union of two closed intervals. .... True

Problem 3

a. One can use the fact that, if $|f - f_n| < \varepsilon$ in $[0, 1]$, and m and M are lower and upper Riemann sums for f_n, then $m - \varepsilon$ and $M + \varepsilon$ will be lower and upper Riemann sums for f.

Alternatively, let each f_n be continuous except on a set $E_n \subset [0, 1]$ of measure zero. Then f will be continuous outside the union of these sets, which also has measure zero. Also, f will be bounded. .... True

b. The points of discontinuity of V is ∂V (the boundary of V). There are open sets whose boundary have positive measure. For example, if $\langle q_n \rangle_{n\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$ then

$$V = \bigcup_{n\in\mathbb{N}} (q_n - 2^{-n}, q_n + 2^n)$$

is dense in $\mathbb{R}$ (since it contains $\mathbb{Q}$), so $\partial V = \mathbb{R} \setminus V$, which has infinite measure, since V has finite measure. .... False

Problem 4

a. $E_{i,j} \subseteq \bigcup_{i\in\mathbb{N}} E_{i,j}$, so $\bigcap_{j\in\mathbb{N}} E_{i,j} \subseteq \bigcap_{j\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} E_{i,j}$. Now take the union over all j on the left. .... True

b. Let $0 \in E_{i,j}$ when $i = j$ and not otherwise. Then 0 belongs to the left set but not the right. ..... False

Problem 5

a. $\liminf_{n\to\infty} f_n(x) < a$ if and only if there is some n so that $f_j(x) < a - 1/n$ infinitely often; i.e., for arbitrarily large j .... F

b. $\liminf_{n\to\infty} f_n(x) \le a$ if and only if $f_j(x) < a + 1/n$ infinitely often, for every n .... D

Problem 6

V is a countable disjoint union of bounded, open intervals (the *components* of V). If (α, β) is one of these intervals then $a(x) = \alpha$ and $b(x) = \beta$ whenever $x \in (\alpha, \beta)$. Further, when $x \notin V$ then $a(x) = b(x) = x$, $f(x) = g(x) = 0$. It follows that the restrictions of f and g to $[\alpha, \beta]$ is continuous, so they are continuous on the right in α and on the left in β.

It only remains to check continuity at points x where an infinite number of the components converge to x.

The maximum value of f and g on $[\alpha, \beta]$ is taken in the middle of the interval, $\frac{1}{4}(\beta - \alpha)^2$ and $\frac{1}{4}$ respectively.

a. Given $x \notin V$, $f(x) = 0$, and for any $\varepsilon > 0$, f takes values $\ge \varepsilon$ only in a finite number of components of V. .... True

b. Follows from **a**. .... True

c. Follows from **b**. .... True

d. Noting that g takes the value $\frac{1}{4}$ at the center of any component, any open set with a boundary point which is not an endpoint of a component provides a counterexample. (E.g., the union of the intervals $(1/(2k+1), 1/(2k))$.) .... False

e. Any open set whose boundary has nonzero measure provides a counterexample (there are only countably many endpoints of components). See Problem **3b**. .... False

f. $\{x \colon g(x) > a\}$ is an open set for all a. .... True

Problem 7

The result on partial integration for the Riemann–Stieltjes integral with $g(x) = x$ yields

$$\int_a^b x\,df(x) = bf(b) - af(a) - I.$$

Problem 8

The result is obviously true if $\theta A_n = \infty$ for any n. So we may assume $\theta A_n < \infty$ for all n. There is a Lebesgue measurable $B_n \supseteq A_n$ with $\mu B_n = \theta A_n$. $B_n \setminus B_{n+1}$ must have measure zero, for else $B_n \cap B_{n+1}$ is a superset of A_n with measure smaller than θA_n. So we can put $C_n = B_0 \cup B_1 \cup B_2 \cup \cdots \cup B_n$ and still have $A_n \subseteq C_n$, $\mu C_n = \theta A_n$, and $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$. But then

$$\theta \bigcup_{n\in\mathbb{N}} A_n \le \mu \bigcup_{n\in\mathbb{N}} C_n = \lim_{n\to\infty} \mu C_n = \lim_{n\to\infty} \theta A_n$$

and the opposite inequality is obvious. . . . . True

Problem 9

A simple subsitution in the (Riemann!) integral yields

$$\left|\int_{\sqrt{(n-1)\pi}}^{\sqrt{n\pi}} \sin(x^2)\,dx\right| = \left|\int_{(n-1)\pi}^{n\pi} \frac{\sin t}{2\sqrt{t}}\,dt\right| > \frac{1}{2\sqrt{n\pi}}\int_{(n-1)\pi}^{n\pi} |\sin t|\,dt = \frac{1}{\sqrt{n\pi}}$$

But the sum of these is infinite, so $x \mapsto |\sin(x^2)|$ cannot be integrable. . . . . False

Problem 10

a. Let $A = [0,1] \cup [2,3]$ and $B = [1,2]$. Then $\psi A = \psi B = 1$, but $\psi(A \cup B) = 3 > \psi A + \psi B$. . . . . False

b. The axioms are easily checked. . . . . True

But this outer measure does not produce an interesting measure (it has too few measurable sets to be useful). For that, you have to put an upper limit δ on the diameter of the balls, and then let $\delta \to 0$. The result is one-dimensional *Hausdorff measure*, which is beyond the scope of this course.

Problem 11

a. The requirement that $\mu\{x : |f(x)| \ge \varepsilon\} < \infty$ for all $\varepsilon > 0$ is missing. And a counterexample can indeed be given as the characteristic function of an infinite atom. . . . . False

b. First, let $F_m = \{x \in X : |f(x)| \ge m\}$. Integrating the inequality $f \ge m\chi F_m$ we get $\int |f|\,d\mu \ge m\mu F_m$. In particular $\mu F_m \to 0$ when $m \to \infty$, and since $F_{m+1} \subseteq F_m$ we see that $f(x)\chi F_m(x) \to 0$ for x outside $\bigcap_{m\in\mathbb{N}} F_m$, which has measure zero. Using the Lebesgue dominated convergence theorem, we conclude $\int_{F_m} |f|\,d\mu \to 0$. Let $\varepsilon > 0$ and pick m large enough so $\int_{F_m} |f|\,d\mu < \varepsilon$. We find

$$\left|\int_{E_n} f\,d\mu\right| \le \int_{E_n} |f|\,d\mu = \int_{E_n\cap F_m} |f|\,d\mu + \int_{E_n\setminus F_m} |f|\,d\mu \le \int_{F_m} |f|\,d\mu + \int_{E_n\setminus F_m} m\,d\mu < \varepsilon + m\mu E_n < 2\varepsilon$$

if n is big enough. (This was harder than intended.) . . . . True

c. If not, then $m := \sup\{\mu B : B \subseteq A, \mu B < \infty\} < \infty$. Pick measurable $B_n \subseteq A$ with $\mu B_n \to m$.
Now $A \setminus \bigcup_{n\in\mathbb{N}} B_n$ would be an infinite atom. . . . . True

d. For a counterexample, take an uncountable set with the counting measure. False

e. If $\mu\{x : |f(x)| \ge \varepsilon\} = \infty$ then there are measurable sets E of finite but arbitrarily large measure so that $|f(x)| \ge \varepsilon$ for $x \in E$. So $\varepsilon\chi E$ is a simple function which is $\le |f|$, and its integral is $\varepsilon\mu E$, which is not bounded above. . . . . True