Candidate no.: ______________

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"Midterm" test, TMA4255 Foundations of Analysis

Thursday, 28 October 2011
Time: approx. 08:15 – 10:00; total 90 minutes

Permitted support materials (code D): Approved calculator (Citizen SR-270X or HP 30S)
This test is only available in English

Instructions: Answer the questions by filling in the boxes provided on these sheets. Corrections are okay so long as the result is unambiguous. Do not include any extra sheets, and do not write in justifications of your answers. Write your candidate number in the upper right corner *on every page* before turning them in.

With every "true or false" question, check False if there exists an example consistent with the assumptions of the problem that makes the statement false.

Problem 1 True or false? Check the correct answer.

a. Assume that $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence of closed subsets of $\mathbb{R}$.
Then $\{x \colon x \in F_n \text{ for all } n, \text{ and } |x| \le 100\}$ is compact. . . . . ☑ True ☐ False

b. Assume that $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence of closed subsets of $\mathbb{R}$.
Then $\{x \colon x \in F_n \text{ for infinitely many } n, \text{ and } |x| \le 100\}$ is compact.
. . . . ☐ True ☑ False

Problem 2 True or false? Check the correct answer.

a. Every disjoint set of closed subsets of $\mathbb{R}$ is countable. . . . . ☐ True ☑ False

b. Every disjoint set of open subsets of $\mathbb{R}$ is countable. . . . . . ☑ True ☐ False

c. Every compact subset of $\mathbb{R}$ can be written as $\bigcap_{n\in\mathbb{N}} F_n$, where each F_n is a finite union of closed intervals. . . . . ☑ True ☐ False

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Problem 3 True or false? Check the correct answer.

a. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of Riemann integrable functions defined on $[0,1]$. If f is a function on $[0,1]$ so that $f_n \to f$ uniformly on $[0,1]$, then f is Riemann integrable. . . . . ☑ True ☐ False

b. The characteristic function of any open set is Riemann integrable over any bounded interval. . . . . ☐ True ☑ False

Problem 4 True or false? Check the correct answer.

Let $\langle E_{i,j} \rangle_{i\in\mathbb{N},j\in\mathbb{N}}$ be any doubly indexed family of sets.

$\bigcup_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} E_{i,j} \subseteq \bigcap_{j\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} E_{i,j}$ . . . . ☑ True ☐ False

$\bigcap_{j\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} E_{i,j} \subseteq \bigcup_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} E_{i,j}$ . . . . ☐ True ☑ False

Problem 5 Let $f_n \colon X \to \mathbb{R}$ for $n = 1, 2, 3, \ldots$. Consider the sets

$$A = \bigcap_{k=1}^{\infty} \bigcup_{j>k} \Big\{ x \in X \colon f_j(x) < a \Big\}$$

$$B = \bigcap_{k=1}^{\infty} \bigcup_{j>k} \Big\{ x \in X \colon f_j(x) \le a \Big\}$$

$$C = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{j>k} \Big\{ x \in X \colon f_j(x) < a - \frac{1}{n} \Big\}$$

$$D = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{j>k} \Big\{ x \in X \colon f_j(x) < a + \frac{1}{n} \Big\}$$

$$E = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{j>k} \Big\{ x \in X \colon f_j(x) < a + \frac{1}{n} \Big\}$$

$$F = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{j>k} \Big\{ x \in X \colon f_j(x) < a - \frac{1}{n} \Big\}$$

Check the correct answers, placing multiple marks if you think several answers are correct:

a. $\{x \in X \colon \liminf_{n\to\infty} f_n(x) < a\}$ equals

☐ A ☐ B ☐ C ☐ D ☐ E ☑ F ☐ none of these

b. $\{x \in X \colon \liminf_{n\to\infty} f_n(x) \le a\}$ equals

☐ A ☐ B ☐ C ☑ D ☐ E ☐ F ☐ none of these

Problem 6 True or false? Check the correct answer.

Let $V \subset \mathbb{R}$ be a bounded, open set. Define functions $\mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} a(x) &= \sup\{y \in \mathbb{R} \setminus V : y \le x\} \\ b(x) &= \inf\{y \in \mathbb{R} \setminus V : y \ge x\} \\ f(x) &= (x - a(x))(b(x) - x) \\ g(x) &= \begin{cases} \dfrac{(x-a(x))(b(x)-x)}{(b(x)-a(x))^2} & x \in V \\ 0 & x \in \mathbb{R} \setminus V \end{cases} \end{aligned}$$

a. f is continuous. . . . . ☑ True ☐ False

b. f is Riemann integrable over any bounded interval. . . . . . ☑ True ☐ False

c. f is Lebesgue integrable. . . . . ☑ True ☐ False

d. g is continuous. . . . . ☐ True ☑ False

e. g is Riemann integrable over any bounded interval. . . . . . ☐ True ☑ False

f. g is Lebesgue integrable. . . . . ☑ True ☐ False

Problem 7 Assume that f is Riemann integrable over $[a, b]$, and

$$\int_a^b f(x)\,dx = I.$$

Fill in the value of the following integral expressed by a, b, I, and a finite number of values of f – or write "impossible" if the integral cannot be so expressed or might not exist:

$$\int_a^b x\,df(x) = \boxed{b\,f(b) - a\,f(a) - I}$$

Problem 8 True or false?

Let θ be Lebesgue outer measure on $\mathbb{R}$, and assume $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is a sequence of (possibly non-measurable) subsets of $\mathbb{R}$. Then

$\theta \bigcup_{n \in \mathbb{N}} A_n = \lim_{n \to \infty} \theta A_n$ . . . . ☑ True ☐ False

Problem 9 True or false? You may take as given that the limit

$$\lim_{m \to \infty} \int_0^m \sin(x^2)\,dx$$

exists. The function $x \mapsto \sin(x^2)$ is Lebesgue integrable on $\mathbb{R}$ . . ☐ True ☑ False

Problem 10 True or false?

a. For any subset $A \subseteq \mathbb{R}$, define

$$\psi A = \inf \sup_{j \in \mathbb{N}} \lambda J_j$$

where λJ_j is the length of J_j, and the infimum is over all disjoint sequences $\langle J_j \rangle_{j \in \mathbb{N}}$ of open intervals so that $A \subseteq \bigcup_j J_j$, Then ψ is an outer measure on $\mathbb{R}$.
. . . . ☐ True ☑ False

b. For any subset $A \subseteq \mathbb{R}^2$, define

$$\varphi A = \inf \sum_{j \in \mathbb{N}} \operatorname{diam} J_j$$

where $\operatorname{diam} J_j$ is the diameter of J_j, and the infimum is over all sequences $\langle J_j \rangle_{j \in \mathbb{N}}$ of open circular disks so that $A \subseteq \bigcup_j J_j$. Then φ is an outer measure on $\mathbb{R}^2$. . . . . ☑ True ☐ False

Problem 11 True or false?

Let (X, Σ, μ) be a measure space.

a. A measurable function f on X is integrable if there is some $M \in \mathbb{R}$ so that $\int g \, d\mu < M$ whenever g is a simple function with $g \le |f|$ a.e.
. . . . ☐ True ☑ False

b. Assume that f be an integrable function on X, and that $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence of measurable sets with $\mu E_n \to 0$ as $n \to \infty$.
Then $\int_{E_n} f \, d\mu \to 0$ as well. . . . . ☑ True ☐ False

We say that a set $A \subseteq X$ is an *infinite atom* if it measurable with $\mu A = \infty$, and for every measurable $B \subseteq A$ either $\mu B = 0$ or $\mu B = \infty$.

In **b–d** below, assume that there are no infinite atoms in X.

c. If $A \in \Sigma$ and $\mu A = \infty$ then for each $M \in \mathbb{R}$ there is some measurable $B \subset A$ with $M < \mu B < \infty$ . . . . ☑ True ☐ False

d. Every $A \in \Sigma$ is a countable union of measurable sets of finite measure.
. . . . ☐ True ☑ False

e. A measurable function f on X is integrable if there is some $M \in \mathbb{R}$ so that $\int g \, d\mu < M$ whenever g is a simple function with $g \le |f|$ a.e.
. . . . ☑ True ☐ False