



Contact during the test:
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“Midterm” test, TMA4255 Foundations of Analysis

Thursday, 28 October 2011

Time: approx. 08:15 – 10:00; total 90 minutes

Permitted support materials (code D): Approved calculator (Citizen SR-270X or HP 30S)

This test is only available in English

Instructions: Answer the questions by filling in the boxes provided on these sheets. Corrections are okay so long as the result is unambiguous. Do not include any extra sheets, and do not write in justifications of your answers. Write your candidate number in the upper right corner *on every page* before turning them in.

With every “true or false” question, check False if there exists an example consistent with the assumptions of the problem that makes the statement false.

Problem 1 True or false? Check the correct answer.

- a. Assume that $\langle F_n \rangle_{n \in \mathbb{N}}$ is a sequence of closed subsets of \mathbb{R} .
Then $\{x: x \in F_n \text{ for all } n, \text{ and } |x| \leq 100\}$ is compact. True False
- b. Assume that $\langle F_n \rangle_{n \in \mathbb{N}}$ is a sequence of closed subsets of \mathbb{R} .
Then $\{x: x \in F_n \text{ for infinitely many } n, \text{ and } |x| \leq 100\}$ is compact.
..... True False

Problem 2 True or false? Check the correct answer.

- a. Every disjoint set of closed subsets of \mathbb{R} is countable. True False
- b. Every disjoint set of open subsets of \mathbb{R} is countable. True False
- c. Every compact subset of \mathbb{R} can be written as $\bigcap_{n \in \mathbb{N}} F_n$, where each F_n is a finite union of closed intervals. True False

Problem 3 True or false? Check the correct answer.

- a. Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a sequence of Riemann integrable functions defined on $[0, 1]$. If f is a function on $[0, 1]$ so that $f_n \rightarrow f$ uniformly on $[0, 1]$, then f is Riemann integrable. True False
- b. The characteristic function of any open set is Riemann integrable over any bounded interval. True False

Problem 4 True or false? Check the correct answer.

Let $\langle E_{i,j} \rangle_{i \in \mathbb{N}, j \in \mathbb{N}}$ be any doubly indexed family of sets.

$$\bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} E_{i,j} \subseteq \bigcap_{j \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} E_{i,j} \dots\dots\dots \input type="checkbox"/> True False$$

$$\bigcap_{j \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} E_{i,j} \subseteq \bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} E_{i,j} \dots\dots\dots \input type="checkbox"/> True False$$

Problem 5 Let $f_n: X \rightarrow \mathbb{R}$ for $n = 1, 2, 3, \dots$. Consider the sets

$$A = \bigcap_{k=1}^{\infty} \bigcup_{j>k} \left\{ x \in X : f_j(x) < a \right\}$$

$$B = \bigcap_{k=1}^{\infty} \bigcup_{j>k} \left\{ x \in X : f_j(x) \leq a \right\}$$

$$C = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{j>k} \left\{ x \in X : f_j(x) < a - \frac{1}{n} \right\}$$

$$D = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{j>k} \left\{ x \in X : f_j(x) < a + \frac{1}{n} \right\}$$

$$E = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{j>k} \left\{ x \in X : f_j(x) < a + \frac{1}{n} \right\}$$

$$F = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{j>k} \left\{ x \in X : f_j(x) < a - \frac{1}{n} \right\}$$

Check the correct answers, placing multiple marks if you think several answers are correct:

- a. $\{x \in X : \liminf_{n \rightarrow \infty} f_n(x) < a\}$ equals

A B C D E F none of these

- b. $\{x \in X : \liminf_{n \rightarrow \infty} f_n(x) \leq a\}$ equals

A B C D E F none of these

Problem 10 True or false?

- a. For any subset $A \subseteq \mathbb{R}$, define

$$\psi A = \inf \sup_{j \in \mathbb{N}} \lambda J_j$$

where λJ_j is the length of J_j , and the infimum is over all disjoint sequences $\langle J_j \rangle_{j \in \mathbb{N}}$ of open intervals so that $A \subseteq \bigcup_j J_j$. Then ψ is an outer measure on \mathbb{R} .

..... True False

- b. For any subset $A \subseteq \mathbb{R}^2$, define

$$\varphi A = \inf \sum_{j \in \mathbb{N}} \text{diam } J_j$$

where $\text{diam } J_j$ is the diameter of J_j , and the infimum is over all sequences $\langle J_j \rangle_{j \in \mathbb{N}}$ of open circular disks so that $A \subseteq \bigcup_j J_j$. Then φ is an outer measure on \mathbb{R}^2 .

..... True False

Problem 11 True or false?

Let (X, Σ, μ) be a measure space.

- a. A measurable function f on X is integrable if there is some $M \in \mathbb{R}$ so that $\int g d\mu < M$ whenever g is a simple function with $g \leq |f|$ a.e.

..... True False

- b. Assume that f be an integrable function on X , and that $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence of measurable sets with $\mu E_n \rightarrow 0$ as $n \rightarrow \infty$.

Then $\int_{E_n} f d\mu \rightarrow 0$ as well. True False

We say that a set $A \subseteq X$ is an *infinite atom* if it measurable with $\mu A = \infty$, and for every measurable $B \subseteq A$ either $\mu B = 0$ or $\mu B = \infty$.

In **b–d** below, assume that there are no infinite atoms in X .

- c. If $A \in \Sigma$ and $\mu A = \infty$ then for each $M \in \mathbb{R}$ there is some measurable $B \subset A$ with $M < \mu B < \infty$

True False

- d. Every $A \in \Sigma$ is a countable union of measurable sets of finite measure.

..... True False

- e. A measurable function f on X is integrable if there is some $M \in \mathbb{R}$ so that $\int g d\mu < M$ whenever g is a simple function with $g \leq |f|$ a.e.

..... True False