# Littlewood's three principles

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**Littlewood's three principles** for the Lebesgue measure on $\mathbb{R}$ are, roughly speaking:

1. Every measurable set of finite measure is nearly a finite union of intervals
2. Every measurable function is nearly continuous
3. Every convergent sequence of functions is nearly uniformly convergent

The second and third principles are also known as *Lusin's* and *Egorov's* theorems, respectively. The first principle has no other name that I know of:

**Theorem 1** (Littlewood's first principle)**.** *Let $E$ be a Lebesgue measurable subset of $\mathbb{R}$, and assume $\varepsilon > 0$ is given. Then there exists a finite union $F$ of intervals such that $\mu(E \triangle F) < \varepsilon$.*

*Proof.* We go back to the definition of Lebegue outer measure: There is a sequence of intervals $\langle I_j \rangle_{j\in\mathbb{N}}$ such that

$$E \subseteq \bigcup_{j\in\mathbb{N}} I_j \quad \text{and} \quad \sum_{j\in\mathbb{N}} \mu I_j < \mu E + \varepsilon.$$

In particular, the sum above converges, so we can fix some $n$ with

$$\sum_{j>n} \mu I_j < \varepsilon.$$

Let

$$F = \bigcup_{j\le n} I_j.$$

Then

$$\mu(F \setminus E) \le \mu\Big(\bigcup_{j\in\mathbb{N}} I_j \setminus E\Big) \le \sum_{j\in\mathbb{N}} \mu I_j - \mu E < \varepsilon$$

and

$$\mu(E \setminus F) \le \mu\Big(\bigcup_{j\in\mathbb{N}} I_j \setminus F\Big) \le \mu \bigcup_{j>n} I_j \le \sum_{j>n} \mu I_j < \varepsilon,$$

so that $\mu(E \triangle F) \le \mu(E \setminus F) + \mu(F \setminus E) < 2\varepsilon$ (and hence the overly pedantic person would replace $\varepsilon$ by $\varepsilon/2$ throughout the proof). □

Littlewood's third principle (Egorov's theorem) is different from the other two, in that it holds in any measure space:

**Theorem 2** (Egorov; Littlewood's third principle)**.** *Assume that* $(X, \Sigma, \mu)$ *is a measure space with* $\mu X < \infty$*, and that* $\langle f_k \rangle_{k \in \mathbb{N}}$ *is a sequence of measurable, real-valued functions with measurable domains converging pointwise a.e. in* $X$ *to a real-valued function* $f$*. Then, given any* $\varepsilon > 0$*, there is a measurable set* $E \subseteq X$ *so that* $\mu(X \setminus E) < \varepsilon$ *and* $f_k \to f$ *uniformly on* $E$*.*

*Proof.* To simplify the notation, consider $h_k = |f_k - f|$, so that $h_k \geq 0$ and $h_k \to 0$ a.e. on $X$. By assumption, for any natural number $m$

$$\mu\Big(\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \{x\colon x \notin \operatorname{dom} h_k \text{ or } h_k(x) \geq m^{-1}\}\Big) = 0,$$

since pointwise convergence fails on this set. The union in the above expression decreases with $n$, and by finiteness of the measure it follows that the measure of the intersection is the limit of the measure of the union as $n \to \infty$. Thus we can pick $n(m)$ so that

$$\mu\Big(\bigcup_{k \geq n(m)} \{x\colon x \notin \operatorname{dom} h_k \text{ or } h_k(x) \geq m^{-1}\}\Big) < 2^{-m}\varepsilon.$$

Put

$$F = \bigcup_{m \geq 1} \bigcup_{k \geq n(m)} \{x\colon x \notin \operatorname{dom} h_k \text{ or } h_k(x) \geq m^{-1}\}.$$

Then adding the above inequalities we get $\mu F < \varepsilon$. Also, $h_k \to 0$ uniformly on $E := X \setminus F$, for

$$E = \bigcap_{m \geq 1} \bigcap_{k \geq n(m)} \{x \in \operatorname{dom} h_k\colon h_k(x) < m^{-1}\},$$

so for every $m$ and every $k \geq n(m)$ it is true that $E \subseteq \operatorname{dom} h_k$ and $h_k < m^{-1}$ on $E$. □

**Theorem 3** (Lusin; Littlewood's second principle)**.** *Let $f$ be a measurable, real-valued function on an interval $I \subseteq \mathbb{R}$. Then, for any $\varepsilon > 0$, there is a measurable set $E \subseteq I$ so that $\mu(I \setminus E) < \varepsilon$ and $f \upharpoonright E$ is continuous.*

*Proof.* First, assume that $I$ is a bounded interval; say, $I = (a, b)$, and that $f$ is a bounded function. We know that if we put

$$F(x) = \int_a^x f$$

then $F$ is continuous in $I$, and furthermore $F' = f$ a.e. in $I$. Define

$$g_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f = \frac{1}{2h}\big(F(x+h) - F(x-h)\big),$$

then it follows that $g_h$ is continuous, and $g_h \to f$ a.e. when $h \downarrow 0$. By Lusin's theorem, there is a set $E \subseteq I$ with $\mu(I \setminus E) < \varepsilon$ so that $g_{1/n} \to f$ uniformly on $E$ as $n \to \infty$. But then $f \upharpoonright E$ is continuous.

Second, let $I$ remain bounded, but allow $f$ to be unbounded. Since

$$\bigcap_{n \in \mathbb{N}} \{x \in I \colon |f(x)| > n\} = \emptyset,$$

we can pick $n$ large enough so that

$$\mu E_0 < \varepsilon, \quad \text{where } E_0 = \{x \in I \colon |f(x)| > n\}.$$

Apply the first part to the bounded function $f\chi E_0$ (details left to the reader.)

Finally, assume that $I$ is an unbounded interval, which we may take to be open. We can find a disjoint sequence of open, bounded subintervals $I_j \subset I$ so that

$$\mu\Big(I \setminus \bigcup_{j \in \mathbb{N}} I_j\Big) = 0.$$

(We can easily arrange for the set difference to be countable. For example, if $I = (0, \infty)$ we can let $I_j = (j, j+1)$.) Apply the second part to $f \upharpoonright I_j$ for each $j$, finding some measurable $E_j \subseteq I_j$ with $\mu(I_j \setminus E_j) < 2^{-j}\varepsilon$ so that $f \upharpoonright E_j$ is continuous for each $j$. Then

$$E = \bigcup_{j \in \mathbb{N}} E_j$$

has the properties we are looking for (again, details are left to the reader). □

**Application.** The following classical result shows that the Fourier transform of an integrable function vanishes at infinity.

**Lemma 4** (Riemann–Lebesgue)**.** *If $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable then*

$$\lim_{s\to\pm\infty} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx = 0.$$

*Proof.* First, we prove the result when $f = \chi I$ where $I$ is a bounded interval, say $I = (a, b)$. This is trivial indeed:

$$\int_a^b e^{isx}\,dx = \frac{e^{isa} - e^{isb}}{is} \to 0 \quad \text{when } s \to \pm\infty.$$

Just as trivially, the result extends to any finite union of bounded intervals.

Next, consider $f = \chi E$ where $E$ is Lebesgue measurable with finite measure. Let $\varepsilon > 0$. There is a finite union $J$ of bounded intervals so that $\mu(E \triangle J) < \varepsilon$. By the first part of the proof,

$$\left|\int_J e^{isx}\,dx\right| < \varepsilon$$

when $|s|$ is large enough, in which case

$$\left|\int_F e^{isx}\,dx\right| \le \left|\int_J e^{isx}\,dx\right| + \left|\int_{F\triangle J} e^{isx}\,dx\right| < 2\varepsilon,$$

so the result is shown when $f = \chi E$. Next, it extends trivially to any simple function.

Finally, if $f$ is integrable then given $\varepsilon > 0$ there is a simple integrable function $g$ so that

$$\int_{\mathbb{R}} |f - g| < \varepsilon.$$

By what we have just proved,

$$\left|\int_{\mathbb{R}} g(x)e^{isx}\,dx\right| < \varepsilon$$

when $|s|$ is large enough, in which case

$$\left|\int_{\mathbb{R}} f(x)e^{isx}\,dx\right| \le \left|\int_{\mathbb{R}} \big(f(x) - g(x)\big)e^{isx}\,dx\right| + \left|\int_{\mathbb{R}} g(x)e^{isx}\,dx\right| < \int_{\mathbb{R}} |f - g| + \varepsilon < 2\varepsilon,$$

and the proof is finished. □