

Littlewood's three principles

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Littlewood's three principles for the Lebesgue measure on \mathbb{R} are, roughly speaking:

1. Every measurable set of finite measure is nearly a finite union of intervals
2. Every measurable function is nearly continuous
3. Every convergent sequence of functions is nearly uniformly convergent

The second and third principles are also known as *Lusin's* and *Egorov's* theorems, respectively. The first principle has no other name that I know of:

Theorem 1 (Littlewood's first principle). *Let E be a Lebesgue measurable subset of \mathbb{R} , and assume $\varepsilon > 0$ is given. Then there exists a finite union F of intervals such that $\mu(E \Delta F) < \varepsilon$.*

Proof. We go back to the definition of Lebesgue outer measure: There is a sequence of intervals $\langle I_j \rangle_{j \in \mathbb{N}}$ such that

$$E \subseteq \bigcup_{j \in \mathbb{N}} I_j \quad \text{and} \quad \sum_{j \in \mathbb{N}} \mu I_j < \mu E + \varepsilon.$$

In particular, the sum above converges, so we can fix some n with

$$\sum_{j > n} \mu I_j < \varepsilon.$$

Let

$$F = \bigcup_{j \leq n} I_j.$$

Then

$$\mu(F \setminus E) \leq \mu\left(\bigcup_{j \in \mathbb{N}} I_j \setminus E\right) \leq \sum_{j \in \mathbb{N}} \mu I_j - \mu E < \varepsilon$$

and

$$\mu(E \setminus F) \leq \mu\left(\bigcup_{j \in \mathbb{N}} I_j \setminus F\right) \leq \mu \bigcup_{j > n} I_j \leq \sum_{j > n} \mu I_j < \varepsilon,$$

so that $\mu(E \Delta F) \leq \mu(E \setminus F) + \mu(F \setminus E) < 2\varepsilon$ (and hence the overly pedantic person would replace ε by $\varepsilon/2$ throughout the proof). \square

Littlewood's third principle (Egorov's theorem) is different from the other two, in that it holds in any measure space:

Theorem 2 (Egorov; Littlewood's third principle). *Assume that (X, Σ, μ) is a measure space with $\mu X < \infty$, and that $\langle f_k \rangle_{k \in \mathbb{N}}$ is a sequence of measurable, real-valued functions with measurable domains converging pointwise a.e. in X to a real-valued function f . Then, given any $\varepsilon > 0$, there is a measurable set $E \subseteq X$ so that $\mu(X \setminus E) < \varepsilon$ and $f_k \rightarrow f$ uniformly on E .*

Proof. To simplify the notation, consider $h_k = |f_k - f|$, so that $h_k \geq 0$ and $h_k \rightarrow 0$ a.e. on X . By assumption, for any natural number m

$$\mu\left(\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \{x: x \notin \text{dom } h_k \text{ or } h_k(x) \geq m^{-1}\}\right) = 0,$$

since pointwise convergence fails on this set. The union in the above expression decreases with n , and by finiteness of the measure it follows that the measure of the intersection is the limit of the measure of the union as $n \rightarrow \infty$. Thus we can pick $n(m)$ so that

$$\mu\left(\bigcup_{k \geq n(m)} \{x: x \notin \text{dom } h_k \text{ or } h_k(x) \geq m^{-1}\}\right) < 2^{-m} \varepsilon.$$

Put

$$F = \bigcup_{m \geq 1} \bigcup_{k \geq n(m)} \{x: x \notin \text{dom } h_k \text{ or } h_k(x) \geq m^{-1}\}.$$

Then adding the above inequalities we get $\mu F < \varepsilon$. Also, $h_k \rightarrow 0$ uniformly on $E := X \setminus F$, for

$$E = \bigcap_{m \geq 1} \bigcap_{k \geq n(m)} \{x \in \text{dom } h_k: h_k(x) < m^{-1}\},$$

so for every m and every $k \geq n(m)$ it is true that $E \subseteq \text{dom } h_k$ and $h_k < m^{-1}$ on E . \square

Theorem 3 (Lusin; Littlewood's second principle). *Let f be a measurable, real-valued function on an interval $I \subseteq \mathbb{R}$. Then, for any $\varepsilon > 0$, there is a measurable set $E \subseteq I$ so that $\mu(I \setminus E) < \varepsilon$ and $f \upharpoonright E$ is continuous.*

Proof. First, assume that I is a bounded interval; say, $I = (a, b)$, and that f is a bounded function. We know that if we put

$$F(x) = \int_a^x f$$

then F is continuous in I , and furthermore $F' = f$ a.e. in I . Define

$$g_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f = \frac{1}{2h} (F(x+h) - F(x-h)),$$

then it follows that g_h is continuous, and $g_h \rightarrow f$ a.e. when $h \downarrow 0$. By Lusin's theorem, there is a set $E \subseteq I$ with $\mu(I \setminus E) < \varepsilon$ so that $g_{1/n} \rightarrow f$ uniformly on E as $n \rightarrow \infty$. But then $f \upharpoonright E$ is continuous.

Second, let I remain bounded, but allow f to be unbounded. Since

$$\bigcap_{n \in \mathbb{N}} \{x \in I : |f(x)| > n\} = \emptyset,$$

we can pick n large enough so that

$$\mu E_0 < \varepsilon, \quad \text{where } E_0 = \{x \in I : |f(x)| > n\}.$$

Apply the first part to the bounded function $f \chi_{E_0}$ (details left to the reader.)

Finally, assume that I is an unbounded interval, which we may take to be open. We can find a disjoint sequence of open, bounded subintervals $I_j \subset I$ so that

$$\mu\left(I \setminus \bigcup_{j \in \mathbb{N}} I_j\right) = 0.$$

(We can easily arrange for the set difference to be countable. For example, if $I = (0, \infty)$ we can let $I_j = (j, j+1)$.) Apply the second part to $f \upharpoonright I_j$ for each j , finding some measurable $E_j \subseteq I_j$ with $\mu(I_j \setminus E_j) < 2^{-j} \varepsilon$ so that $f \upharpoonright E_j$ is continuous for each j . Then

$$E = \bigcup_{j \in \mathbb{N}} E_j$$

has the properties we are looking for (again, details are left to the reader). \square

Application. The following classical result shows that the Fourier transform of an integrable function vanishes at infinity.

Lemma 4 (Riemann–Lebesgue). *If $f: \mathbb{R} \rightarrow \mathbb{R}$ is Lebesgue integrable then*

$$\lim_{s \rightarrow \pm\infty} \int_{-\infty}^{\infty} f(x) e^{isx} dx = 0.$$

Proof. First, we prove the result when $f = \chi I$ where I is a bounded interval, say $I = (a, b)$. This is trivial indeed:

$$\int_a^b e^{isx} dx = \frac{e^{isa} - e^{isb}}{is} \rightarrow 0 \quad \text{when } s \rightarrow \pm\infty.$$

Just as trivially, the result extends to any finite union of bounded intervals.

Next, consider $f = \chi E$ where E is Lebesgue measurable with finite measure. Let $\varepsilon > 0$. There is a finite union J of bounded intervals so that $\mu(E \Delta J) < \varepsilon$. By the first part of the proof,

$$\left| \int_J e^{isx} dx \right| < \varepsilon$$

when $|s|$ is large enough, in which case

$$\left| \int_F e^{isx} dx \right| \leq \left| \int_J e^{isx} dx \right| + \left| \int_{F \Delta J} e^{isx} dx \right| < 2\varepsilon,$$

so the result is shown when $f = \chi E$. Next, it extends trivially to any simple function.

Finally, if f is integrable then given $\varepsilon > 0$ there is a simple integrable function g so that

$$\int_{\mathbb{R}} |f - g| < \varepsilon.$$

By what we have just proved,

$$\left| \int_{\mathbb{R}} g(x) e^{isx} dx \right| < \varepsilon$$

when $|s|$ is large enough, in which case

$$\left| \int_{\mathbb{R}} f(x) e^{isx} dx \right| \leq \left| \int_{\mathbb{R}} (f(x) - g(x)) e^{isx} dx \right| + \left| \int_{\mathbb{R}} g(x) e^{isx} dx \right| < \int_{\mathbb{R}} |f - g| + \varepsilon < 2\varepsilon,$$

and the proof is finished. \square