Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220 Examination Paper 2025

You may use any results presented within a problem without proof, unless explicitly stated otherwise.

This exam consists of **4** questions over **4** pages.

Problem 1

[Total points: 15]

In this question we focus on the definition, and construction, of finite elements.

Definition: Let

- $K \subset \mathbb{R}^n$ be a closed bounded set with nonempty interior and piece-wise smooth boundary,
- $\mathcal{P}$ be a finite dimensional space of functions on K,
- $\mathcal{N} = \{N_1, N_2, ..., N_k\}$ be a basis for $\mathcal{P}^*$,

then, $(K, \mathcal{P}, \mathcal{N})$ is a finite element (Ciarlet triple).

[5 points] **a)** Give the definition of the quartic (degree 4) Lagrange finite element in 2D. You may define this element pictographically.

Definition: We call $\{x : L(x) = 0\}$ a hyperplane for non-degenerate linear $L(x)$. We will abuse notation by referring to L as the hyperplane.

Deconstruction lemma: Let P be a polynomial of degree $d \geq 1$ vanishing on the hyperplane L. Then we have $P = LQ$, where Q is polynomial of degree $d - 1$.

[10 points] **b)** Prove that the quartic Lagrange elements satisfy Ciarlet's definition of a finite element. Explicitly justify why each condition of the definition is met.

Problem 2
[Total points: 18]

Let $u \in H^2(\Omega)$ solve the Poisson equation

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \subset \mathbb{R}^3 \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1}$$

for some data $f \in L_2(\Omega)$.

[4 points] **a)** Show that the Poisson equation can be written as seeking $u \in \mathbb{V}$ such that

$$a(u, v) = g(v) \qquad \forall v \in \mathbb{V},$$

by finding the bilinear form $a(\cdot, \cdot)$, bounded linear functional $g(\cdot)$ and function space $\mathbb{V}$.

Poincaré inequality: Suppose $\Omega \subset \mathbb{R}^3$ is bounded and $u \in \mathbb{V}$, then there exists a constant C_Ω such that

$$\|u\|_{L_2(\Omega)} \leq C_\Omega \|\nabla u\|_{L_2(\Omega)}.$$

[5 points] **b)** Prove the Poincaré inequality for $u \in \mathbb{V}$.

Hint: Initially assume that $u \in C_0^\infty(\Omega)$ and consider $\nabla \cdot (u^2 x)$.

Lax-Milgram theorem: Let $\mathbb{V}$ be a Hilbert space, $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be $\mathbb{V}$-elliptic, and $g \in \mathbb{V}^*$ be a bounded linear functional. Then, there exists a unique $u \in \mathbb{V}$ such that

$$g(v) = a(u, v) \qquad \forall v \in \mathbb{V}.$$

[9 points] **c)** What does it mean for a bilinear form to be $\mathbb{V}$-elliptic? Prove that the bilinear form you found in part (a) has a unique solution. *Hint: To verify coercivity, you may find the splitting $a(u, v) = \frac{1}{2}a(u, v) + \frac{1}{2}a(u, v)$ useful.*

Problem 3
[Total points: 17]

Let $\mathbf{u} \in [C^1([0, T])]^D$ solve the D-dimensional system of ODEs

$$\mathbf{u_t} + A\mathbf{u} = 0, \tag{2}$$

subject to initial data $\mathbf{u}(0) = \mathbf{u}_0$, where $A \in \mathbb{R}^{D \times D}$ is a constant symmetric positive definite matrix.

[3 points] **a)** What is the variational (strong) formulation for this problem? Justify your choice of function spaces.

Partition the domain $[0,T]$ such that $0 = t_0 < t_1 < ... < t_N = T$ and define an arbitrary element over this domain as $I_n = (t_n, t_{n+1})$. In the remainder of this question we will study the conforming formulation for this problem *(the cG method)*, which is given by seeking $\mathbf{U} \in \mathbb{V}_{cG}^k([0,T])$ such that

$$\int_{I_n} \mathbf{U}_t \cdot \mathbf{V} \mathrm{d}t + \int_{I_n} A\mathbf{U} \cdot \mathbf{V} \mathrm{d}t = 0 \qquad \forall \mathbf{V} \in \mathbb{V}_{dG}^{k-1}([0,T]), \tag{3}$$

where $\mathbf{U}(t_0) := \mathbf{u}_0$.

Interpolation operator: For $\mathbf{w} \in [C^0([0,T]) \cap L^2([0,T])]^D$ define the interpolation operator $\mathcal{I} \in \mathbb{V}_{cG}^k$ such that

$$\int_{I_n} (\mathcal{I}(\mathbf{w}) - \mathbf{w}) \cdot \mathbf{V} \mathrm{d}t = 0 \qquad \forall \mathbf{V} \in \mathbb{V}_{dG}^{k-1}$$

and $\mathcal{I}(\mathbf{w})(t_n) = \mathbf{w}(t_n)$ for $n = 0, ..., N$ and $k \geq 1$.

Convergence of interpolation operator: The operator $\mathcal{I}$ converges optimally, i.e.,

$$\|\mathbf{w} - \mathcal{I}(\mathbf{w})\|_{L^2(I_n)} \leq C\tau^{k+1} |\mathbf{w}|_{H^{k+1}(I_n)},$$

for $C > 0$ if $\mathbf{w} \in H^{k+1}(I_n)$, where $\tau := \max_n (t_{n+1} - t_n)$.

[8 points] **b)** Assuming that A induces a continuous and coercive vector norm, prove that,

$$|\mathbf{U}(t_n) - \mathbf{u}(t_n)|^2 \leq C\tau^{2k+2} |\mathbf{u}|^2_{H^{k+1}([0,t_{n+1}])},$$

where $\mathbf{u}$ solves the continuous problem (2) and $\mathbf{U}$ solves (3).

Hint: Start from the error equation

$$\int_{I_n} (\mathbf{U}_t - \mathbf{u}_t) \cdot \mathbf{V} \mathrm{d}t + \int_{I_n} A(\mathbf{U} - \mathbf{u}) \cdot \mathbf{V} \mathrm{d}t = 0 \qquad \forall \mathbf{V} \in \mathbb{V}_{dG}^{k-1},$$

and split the error with the interpolation operator $\mathcal{I}(\mathbf{u}) \in \mathbb{V}_{cG}^k$.

[6 points] **c)** Consider Crank-Nicolson, i.e., given some initial data $\tilde{\mathbf{u}}_0 := \mathbf{u}(0)$ seek $\tilde{\mathbf{u}}_1$ such that

$$\tilde{\mathbf{u}}_1 = \tilde{\mathbf{u}}_0 - \frac{\tau}{2} A(\tilde{\mathbf{u}}_1 + \tilde{\mathbf{u}}_0).$$

Show this method is equivalent to the cG method (3) when $k = 1$. You may assume the partition is uniform, i.e., that $\tau = t_{n+1} - t_n$ is constant for all n.

Hint 1: When $k = 1$, one may write $U|_{I_n} = U_{n+1}\left(\frac{t-t_n}{\tau}\right) - U_n\left(\frac{t-t_{n+1}}{\tau}\right)$.

Hint 2: For linear polynomials, midpoint quadrature is exact, i.e, for $a, b \in \mathbb{R}$ we have that $\int_{I_n} at + b dt = \tau\left(a t_{n+\frac{1}{2}} + b\right)$.

Problem 4
[Total points: 23]

In this question we consider a nonconforming approximation of the Poisson equation with zero Dirichlet boundary conditions as described in (1). This approximation is described by seeking $u_h \in \mathbb{V}_h(\Omega)$ such that

$$a_h(u_h, v_h) = \langle f, v_h \rangle \qquad \forall v_h \in \mathbb{V}_h(\Omega), \tag{4}$$

where $f \in L_2(\Omega)$ and $\mathbb{V}_h \nsubseteq H_0^1(\Omega)$. In particular, we choose

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{P}_h} \int_T \nabla u_h \cdot \nabla v_h \mathrm{d}x - \sum_{e \in \mathcal{E}_h} \int_e \Big(\{\nabla u_h\} \cdot [\![v_h]\!] + \{\nabla v_h\} \cdot [\![u_h]\!] \Big) \mathrm{d}s + \sum_{e \in \mathcal{E}_h} \frac{\eta}{|e|} \int_e [\![u]\!] \cdot [\![v]\!] \mathrm{d}s,$$

where η is sufficiently big, $\mathcal{P}_h$ is the set of elements in the mesh and $\mathcal{E}_h$ the set of edges over the mesh.

[3 points] **a)** The flux terms are individually referred to as the consistency term, symmetry term and penalty term. Identify which is which and heustically justify the naming convention.

[5 points] **b)** For a function $v_h \in \mathbb{V}_h(\Omega)$ with $\Omega \subset \mathbb{R}^3$, give the definition of the jump $[\![v_h]\!]$ and average $\{v_h\}$.

[7 points] **c)** Prove that the interior penalty bilinear form is consistent. That is to say, apply a function $u \in H^2 \cap H_0^1(\Omega)$ to (4) and recover strong form of the Poisson equation (1) tested with $v_h \in \mathbb{V}_h$.

[3 points] **d)** Using what you have proven in (c), show Galerkin orthogonality.

Mesh-dependent norm: We define the dG norm as

$$\|v_h\|_{dG}^2 := \sum_{T \in \mathcal{P}_h} \|\nabla v_h\|_{L_2(T)}^2 + \eta \sum_{e \in \mathcal{E}_h} |e|^{-1} \|[\![v_h]\!]\|_{L_2(e)}^2$$

Assumption: The bilinear form $a_h : \mathbb{V}_h \times \mathbb{V}_h \to \mathbb{R}$ is continuous and coercive, i.e.,

$$a_h(u_h, v_h) \leq c_0 \|u_h\|_{dG} \|v_h\|_{dG}$$
$$a_h(v_h, v_h) \geq c_1 \|v_h\|_{dG}^2.$$

[5 points] **e)** Prove Cea's lemma, i.e., show for $u \in H^2(\Omega) \cap \mathbb{V}_h$ solving the continuous problem (1) and u_h solving the nonconforming approximation (4) that

$$\|u - u_h\|_{dG} \leq C \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{dG}.$$