Norwegian University of Science and Technology
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TMA4220 Examination Paper 2024

You may use any results presented within a problem without proof, unless explicitly stated otherwise.

Problem 1

[Total points: 12]

Let $u \in H^2(\Omega)$ solve the Poisson equation

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \subset \mathbb{R}^3 \\ \nabla u \cdot \mathbf{n} &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1}$$

for some data $f \in L_2(\Omega)$ where $\mathbf{n}$ is the unit normal vector to $\partial\Omega$.

[4 points] **a)** Show that the Poisson equation can be written as seeking $u \in \mathbb{V}$ such that

$$a(u, v) = g(v) \qquad \forall v \in \mathbb{V}, \tag{2}$$

by finding the bilinear form $a(\cdot, \cdot)$ and bounded linear functional $g(\cdot)$. *Note: You may proceed assuming $\mathbb{V} = H^1$ and neglect issues with the kernel of the bilinear form.*[1]

For the remainder of this question you may assume the following results without proof.
Poincaré inequality: Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $u \in \mathbb{V}$, then there exists a constant C_Ω such that

$$\|u\|_0 \leq C_\Omega \|\nabla u\|_0. \tag{3}$$

Lax-Milgram theorem: Let $\mathbb{V}$ be a Hilbert space, $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be $\mathbb{V}$-elliptic, and $g \in \mathbb{V}^*$ be a bounded linear functional on the dual space of $\mathbb{V}$. Then, there exists a unique $u \in \mathbb{V}$ such that

$$g(v) = a(u, v) \qquad \forall v \in \mathbb{V}. \tag{4}$$

[8 points] **b)** What does it mean for a bilinear form to be $\mathbb{V}$-elliptic? Prove that the bilinear form you found in part (a) has a unique solution. *Hint: To verify coercivity, you may find the splitting $a(u, v) = \frac{1}{2}a(u, v) + \frac{1}{2}a(u, v)$ useful.*

[1] The true function space required for this problem is $H^1(\Omega)/\mathbb{R}$, since the solution can only be determined up to a constant.

Problem 2
[Total points: 19]

In this question we focus on the definition, and construction, of finite elements.

[3 points] **a)** State (Ciarlet's) definition of a finite element.

[5 points] **b)** Give the definition of the cubic Lagrange finite element in 2D. You may define this element pictographically.

For the remainder of this question, you may use the following without proof.

Definition: We call $\{x : L(x) = 0\}$ a hyperplane for non-degenerate $L(x)$. We will abuse notation by referring to L as the hyperplane.

Deconstruction lemma: Let P be a polynomial of degree $d \geq 1$ vanishing on the hyperplane L. Then we have $P = LQ$, where Q is polynomial of degree $d - 1$.

[7 points] **c)** Prove that the cubic Lagrange elements satisfy Ciarlet's definition of a finite element. *Hint: Show that the set of nodal variables fully determine the polynomial space over the element using the deconstruction lemma.*

[4 points] **d)** Draw a small mesh of (at least) four elements. Highlight which degrees of freedom need to be fixed and which need to be shared between elements for the finite elements to form a conforming subspace of H_0^1.

Problem 3
[Total points: 21]

Let $\mathbf{u} \in [C^1([0,T])]^D$ solve the D-dimensional system of ODEs

$$\mathbf{u_t} + A\mathbf{u} = 0, \tag{5}$$

subject to initial data $\mathbf{u}(0) = \mathbf{u}_0$, where $A \in \mathbb{R}^{d\times d}$ is a constant symmetric matrix.

[4 points] **a)** What is the variational (strong) formulation for this problem? Justify your choice of function spaces.

Partition the domain $[0,T]$ such that $0 = t_0 < t_1 < ... < t_N = T$ and define an arbitrary element over this domain as $I_n = (t_n, t_{n+1})$. In the remainder of this equation we will study

the upwind dG formulation for this problem, which is given by seeking $\mathbf{U} \in \mathbb{V}_{dG}^k([0,T])$ such that

$$\sum_{n=0}^{N-1} \int_{I_n} \mathbf{U}_t \cdot \mathbf{V} \mathrm{d}t + [\![\mathbf{U}_n]\!] \cdot \mathbf{V}_n^+ + \int_{I_n} A\mathbf{U} \cdot \mathbf{V} \mathrm{d}t = 0 \qquad \forall \mathbf{V} \in \mathbb{V}_{dG}^k([0,T]), \tag{6}$$

where $\mathbf{U}_0^- := \mathbf{u}_0$ and $[\![\mathbf{U}_n]\!] := \mathbf{U}_n^+ - \mathbf{U}_n^-$.

[5 points] **b)** Prove that, for any $\mathbf{W} \in \mathbb{V}_{dG}^k([0,T])$, that

$$\int_{I_n} \mathbf{W}_t \cdot \mathbf{W} \mathrm{d}t + [\![\mathbf{W}_n]\!] \cdot \mathbf{W}_n^+ \geq \frac{1}{2}|\mathbf{W}_{n+1}^-|^2 - \frac{1}{2}|\mathbf{W}_n^-|^2. \tag{7}$$

Interpolation operator: For $\mathbf{w} \in [C^0([0,T]) \cap L^2([0,T])]^D$ define the interpolation operator $\mathcal{I}$ such that

$$\int_{I_n} (\mathcal{I}(\mathbf{w}) - \mathbf{w}) \cdot \mathbf{V} \mathrm{d}t = 0 \qquad \forall \mathbf{V} \in \mathbb{V}_{dG}^{k-1}$$

and $\mathcal{I}(\mathbf{w})(t_n) = \mathbf{w}(t_n)$ for $n = 0, ..., N-1$ and $k \geq 1$.

Convergence of interpolation operator: The operator $\mathcal{I}$ converges optimally, i.e.,

$$\|\mathbf{w} - \mathcal{I}(\mathbf{w})\|_{L^2(I_n)} \leq C\tau^{k+1}|\mathbf{w}|_{H^{k+1}(I_n)}, \tag{8}$$

for $C > 0$ if $\mathbf{w} \in H^{k+1}(I_n)$, where $\tau := \max_n (t_{n+1} - t_n)$.

[8 points] **c)** Assuming that A induces a continuous and coercive vector norm, prove that,

$$|\mathbf{U}_N^- - \mathbf{u}_N^-|^2 \leq C\tau^{2k+2}|\mathbf{u}|_{H^{k+1}}^2, \tag{9}$$

where $\mathbf{u}$ solves the continuous problem and $\mathbf{U}$ the upwind dG method.

Hint: Start from the error equation

$$\int_{I_n} (\mathbf{U}_t - \mathbf{u}_t) \cdot \mathbf{V} \mathrm{d}t + [\![\mathbf{U}_n - \mathbf{u}_n]\!] \cdot \mathbf{V}_\mathbf{n}^+ + \int_{I_n} A\,(\mathbf{U} - \mathbf{u}) \cdot \mathbf{V} \mathrm{d}t = 0 \qquad \forall \mathbf{V} \in \mathbb{V}_{dG}^k, \tag{10}$$

and split the error with the interpolation operator $\mathcal{I}(\mathbf{u}) \in \mathbb{V}_{dG}^k$.

[4 points] **d)** Consider backward Euler, i.e., given some initial data $\widetilde{\mathbf{u}}_0 := \mathbf{u}(0)$ seek $\widetilde{\mathbf{u}}_1$ such that

$$\widetilde{\mathbf{u}}_1 = \widetilde{\mathbf{u}}_0 - \tau A\widetilde{\mathbf{u}}_1. \tag{11}$$

Show this method is equivalent to the upwind dG method (6) when $k = 0$.

Problem 4
[Total points: 20]

Let $u \in H^1([0,T], H_0^1(\Omega))$ describe the weak solution of the heat equation, i.e.,

$$\begin{aligned} \langle u_t, v\rangle + \langle \nabla u, \nabla v\rangle &= \langle f, v\rangle \qquad \forall v \in H_0^1(\Omega) \\ u(0,x) &= u_0(x) \qquad \text{for } u_0 \in H_0^1(\Omega)' \end{aligned} \tag{12}$$

where $\langle \cdot, \cdot \rangle$ is the $L_2(\Omega)$ inner product over the spatial domain.

[4 points] **a)** Show, for the continuous weak form, that

$$\|u(t)\| \leq \|u_0\| + C \int_0^t \|f(s)\| \mathrm{d}s, \tag{13}$$

and determine the constant C.

Now we restrict to the finite dimensional setting in space, i.e., seek $u_h \in C^1([0,T], \mathbb{V}_h)$ such that

$$\begin{aligned} \langle u_{h,t}, v_h\rangle + \langle \nabla u_h, \nabla v_h\rangle &= \langle f, v_h\rangle \qquad \forall v_h \in \mathbb{V}_h(\Omega) \\ u_h(0,x) &= \Pi_{\mathbb{V}_h} u_0(x), \end{aligned} \tag{14}$$

where $\Pi_{\mathbb{V}_h}$ is the L_2 projection into the space $\mathbb{V}_h$. Note we restrict $\mathbb{V}_h$ such that $\mathbb{V}_h \subset \mathbb{V}$ and $\mathbb{V}_h = \text{span}(\{\phi_i\})$.

[4 points] **b)** Write the underlying system in terms of a mass matrix M, stiffness matrix A, right hand side b and initial data. Write an arbitrary element of each of these matrices and vectors in terms of the basis functions.

[5 points] **c)** Prove that M is positive definite.

[7 points] **d)** Fully discretise the method by discretising in time with backward Euler. Given some initial data U^0, prove there is a unique (fully discrete) solution for the solution U^1 (i.e., show that there is a unique solution after the first time step). *Note: You may use that the stiffness matrix A is positive semi-definite without proof.*