**Norwegian University of Science and Technology**
**Department of Mathematical Sciences**

# TMA4220 Examination Paper 2023

You may use any results presented within a problem without proof, unless explicitly stated otherwise.
Throughout this exam $\langle , \rangle_{\mathbb{V}}$ represents the inner product induced by the Hilbert space $\mathbb{V}$, and $\langle u, v\rangle := \int_\Omega u \cdot v \mathrm{d}x$ is the $L_2$ inner product.

## Problem 1

**[Total points: 24]**

Let $u \in H^2(\Omega)$ solve the Poisson equation

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \subset \mathbb{R}^3 \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1}$$

for some data $f \in L_2(\Omega)$.

**[4 points]** **a)** Define the function spaces $L_2(\Omega)$ and $H^2(\Omega)$. Here you may use the weak derivative operator $D^\alpha_w$ without defining it.

**[5 points]** **b)** Show that the Poisson equation can be written as seeking $u \in H^1_0(\Omega)$ such that

$$a(u, v) = g(v) \qquad \forall v \in H^1_0(\Omega), \tag{2}$$

by finding the bilinear form $a(\cdot, \cdot)$ and bounded linear functional $g(v)$.

For the remainder of this question you may assume the following results without proof.
*Poincaré inequality:* Suppose $\Omega \in \mathbb{R}^n$ is bounded and $u \in H^1_0$, then there exists a constant $C$ such that

$$\|u\|_{L_2(\Omega)} \leq C_\Omega \|\nabla u\|_{L_2(\Omega)}. \tag{3}$$

*Riesz representation theorem:* Let $\mathbb{V}$ be a Hilbert space and $g \in \mathbb{V}^*$ be a bounded linear functional on the dual space of $\mathbb{V}$. Then, there exists a unique $u \in \mathbb{V}$ such that

$$g(v) = \langle u, v\rangle_{\mathbb{V}} \qquad \forall v \in \mathbb{V}. \tag{4}$$

**[7 points]** **c)** Prove that the bilinear form you found in part (b) is elliptic in $H^1_0$ and symmetric. If you could not complete part (b) choose any non-degenerate bilinear form.

**[8 points]** **d)** Prove for any symmetric and $\mathbb{V}$-elliptic bilinear form $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ there is a unique solution to (2). Is the weak form you found in (b) well posed?

You may *not* use the Lax-Milgram theorem without proof to answer this question.

## Problem 2

**[Total points: 30]**

In this question we study Lagrange finite elements and their application to a particular weak formulation. We begin by studying the definition of a finite element.
*Definition:* Let

- $K \subset \mathbb{R}^n$ be a closed bounded set with nonempty interior and piece-wise smooth boundary,
- $\mathcal{P}$ be a finite dimensional space of functions on $K$,
- $\mathcal{N} = \{N_1, N_2, ..., N_k\}$ be a basis for $\mathcal{P}^*$,

then, $(K, \mathcal{P}, \mathcal{N})$ is a finite element (Ciarlet triple).

**[2 points]** **a)** Remark on one thing which is required in addition to Ciarlet's definition of a finite element to implement a finite element method.

**[5 points]** **b)** Define the quadratic Lagrange finite element for a triangular mesh. This may be done through sketching the element over an arbitrary triangle and describing the sketch.

*Definition:* $\mathcal{N}$ determines $\mathcal{P}$ if for $\psi \in \mathcal{P}$ with $N(\psi) = 0 \ \forall N \in \mathcal{N}$ we have $\psi = 0$.

*Definition:* We call $\{x : L(x) = 0\}$ a hyperplane for non-degenerate $L(x)$. We will abuse notation by referring to $L$ as the hyperplane.

*Deconstruction lemma:* Let $P$ be a polynomial of degree $d \geq 1$ vanishing on the hyperplane $L$. Then we have $P = LQ$, where $Q$ is polynomial of degree $d - 1$.

**[5 points]** **c)** Prove the quadratic Lagrange element satisfies Ciarlet's definition of a finite element.

Now we shall study the convergence of a finite element approximation using quadratic Lagrange elements. Consider the weak formulation given by seeking $u \in H_0^1(\Omega)$ such that

$$\langle \nabla u, \nabla v \rangle + \langle u, v \rangle = \langle f, v \rangle \qquad \forall v \in H_0^1(\Omega). \tag{5}$$

**[3 points]** **d)** Assuming $u \in H_0^1 \cap H^2$, find the strong form of (5). Write down the pointwise form of the equation and associated boundary conditions.

*Lax-Milgram theorem:* Let $\mathbb{V}$ be a Hilbert space, $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be $\mathbb{V}$-elliptic, and $g : \mathbb{V} \to \mathbb{R}$ be a bounded linear functional on $\mathbb{V}$. Then, there exists a unique $u \in \mathbb{V}$ such that $a(u, v) = g(v)$ for all $v \in \mathbb{V}$.

**[6 points]** **e)** Formulate (5) over a finite dimensional subspace $\mathbb{V}_h \subset \mathbb{V} = H_0^1$ and prove that the discretisation has a unique solution.

*Cea's lemma:* If $u_h \in \mathbb{V}_h$ is the finite element approximation found in (e) and $u \in H_0^1$ solves the weak form (5), then

$$\|u - u_h\|_{H_0^1(\Omega)} \leq C \min_{v_h \in \mathbb{V}_h} \|u - v_h\|_{H_0^1(\Omega)}. \tag{6}$$

**[5 points]** **f)** Prove Cea's lemma and determine the constant $C$.

*Accuracy of Lagrange finite elements:* Let $\{\mathcal{T}_h\}$ for $0 < h \leq 1$ be a nondegenerate family of triangulations for $\Omega \in \mathbb{R}^n$, and let $\mathbb{V}_h$ be the space of degree $k$ finite elements where $k+1-\frac{n}{2} > 0$ and $\mathcal{I} : C^0(\Omega) \to \mathbb{V}_h$ is the interpolant. Then, there exists a constant $C$, independent of $h$, such that if $u \in H^{k+1}$ then

$$\left( \sum_{T \in \mathcal{T}_h} \|u - \mathcal{I}u\|_{H^s(T)}^2 \right)^{\frac{1}{2}} \leq C h^{k+1-s} |u|_{H^{k+1}(\Omega)} \tag{7}$$

for $0 \leq s \leq k+1$.

**[4 points]** **g)** If $\mathbb{V}_h$ is spanned by quadratic Lagrange elements over a family of meshes $\{\mathcal{T}_h\}$ satisfying the above theorem, prove the order of convergence of the approximation found in part (e) when $u \in H^3(\Omega)$.

## Problem 3

**[Total points: 20]**

In this question we consider the spatially discrete, temporally continuous heat equation. To be more concise, seek $u_h \in C^1([0,T], \mathbb{V}_h)$ such that

$$\begin{aligned} \langle u_{h,t}, v_h \rangle + \langle \nabla u_h, \nabla v_h \rangle &= \langle f, v_h \rangle \qquad \forall v_h \in \mathbb{V}_h(\Omega) \\ u_h(0,x) &= \Pi_{\mathbb{V}_h} u_0(x), \end{aligned} \tag{8}$$

where $\Pi_{\mathbb{V}_h} : H^1(\Omega) \to \mathbb{V}_h$ is the $L_2$ projection and $\mathbb{V}_h \subset H_0^1(\Omega)$. Here we shall study the convergence of the spatially discrete problem, which requires the following:

*Ritz projection:* The Ritz (or elliptic) projection is defined by $\mathcal{R}_h : H_0^1 \to \mathbb{V}_h$ such that for any $v \in \mathbb{V}_h$

$$a(\mathcal{R}_h v - v, \chi) = 0 \qquad \forall \chi \in \mathbb{V}_h, \tag{9}$$

where $a(\cdot,\cdot)$ is the bilinear form induced by the elliptic contribution of (8), namely

$$a(v, \chi) = \langle \nabla v, \nabla \chi \rangle .$$

*Regularity assumption on initial data:* Let $\Omega \subset \mathbb{R}^n$ and assume we have a family of regular triangulations $\{\mathcal{T}_h\}$ with $\mathcal{T}_h \subset H_0^1(\Omega)$ such that, for some $r \geq 2$ and small enough $h$

$$\inf_{\chi \in \mathbb{V}_h} \{\|u_0 - \chi\|_{L_2(\Omega)} + h\|\nabla(u_0 - \chi)\|_{L_2(\Omega)}\} \leq Ch^s |u_0|_{H^s(\Omega)} \text{ for } 1 \leq s \leq r, \tag{10}$$

when $u_0 \in H^s \cap H_0^1$ with $u_0$ as described in (8).

[5 points] **a)** Prove that the Ritz projection (9) is stable, i.e., that

$$\|\nabla \mathcal{R}_h v\|_{L_2(\Omega)} \leq \|\nabla v\|_{L_2(\Omega)}, \tag{11}$$

for $v \in H_0^1(\Omega)$.

*Convergence of the Ritz projection in the energy norm:* For $u_0 \in H^s \cap H_0^1(\Omega)$, the Ritz projection converges optimally with respect to the initial data in the $H^1$ semi-norm, i.e.,

$$\|\nabla(u_0 - \mathcal{R}u_0)\|_{L_2(\Omega)} \leq Ch^{s-1}|u_0|_{H^s(\Omega)}. \tag{12}$$

*Elliptic regularity:* Consider the dual problem $-\Delta\psi = \varphi$ for $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$, with $\varphi \in L^2(\Omega)$. Then, there exists a $C > 0$ such that

$$\|\psi\|_{H^2(\Omega)} \leq C_k \|\varphi\|_{L_2(\Omega)}. \tag{13}$$

[7 points] **b)** Prove that, when applied to smooth enough functions, the Ritz projection converges optimally in the $L_2$ norm, i.e., that

$$\|u_0 - \mathcal{R}u_0\|_{L_2(\Omega)} \leq Ch^s \|u_0\|_{H^s(\Omega)}. \tag{14}$$

This proof requires the use of elliptic regularity, to get started consider $\langle \mathcal{R}u_0 - u_0, \varphi \rangle$ where $\varphi \in L_2(\Omega)$, apply the definition of $\varphi$ and exploit elliptic regularity.

[8 points] **c)** Through splitting the error into elliptic and parabolic parts by defining

$$u_h(t) - u(t) = u_h(t) - \mathcal{R}_h u(t) + \mathcal{R}_h u(t) - u(t) =: \theta(t) + \rho(t),$$

prove *one* of the following two results:

1. Let $u \in C^1([0,T], H^s(\Omega))$ solve the heat equation and $u_h$ be the finite dimensional approximation given in (8). Further assume that the regularity assumption on the initial data is satisfied, then

$$\|u_h(t) - u(t)\|_{L_2(\Omega)} \leq \|\Pi_{\mathbb{V}_h} u_0 - u_0\|_{L_2(\Omega)} + Ch^s \left( \|u_0\|_{H^s(\Omega)} + \int_0^t \|u_\tau\|_{L_2(\Omega)} \mathrm{d}\tau \right). \tag{15}$$

2. Let $u \in C^1([0,T], H^s(\Omega))$ solve the heat equation and $u_h$ be the finite dimensional approximation given in (8). Further assume that the regularity assumption on the initial data is satisfied, then

$$\begin{aligned}\|\nabla(u_h(t) - u(t))\|^2_{L_2(\Omega)} \leq C\|\nabla(\Pi_{\mathbb{V}_h} u_0 - u_0)\|^2_{L_2(\Omega)} \\ + Ch^{2s-2}\Bigg(\|u_0\|^2_{H^s(\Omega)} + \|u\|^2_{H^s(\Omega)} \\ + \int_0^t \|u_\tau\|^2_{H^{s-1}(\Omega)} \mathrm{d}\tau\Bigg). \quad (16)\end{aligned}$$