Norwegian University of Science
and Technology
Department of Mathematical
Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Fall 2023

Solutions to exercise set 1

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$$\begin{cases} -u''(x) + c(x)u(x) = f(x), \ \in (a,b) \\ u'(a) + \alpha u(a) = 1, \ \alpha \in \mathbb{R} \\ u(b) = 0, \end{cases} \tag{1}$$

$$\begin{cases} -u''(x) + c(x)u(x) = f(x), \quad x \in (a,b) \\ u(a) = u(b) = 0, \end{cases} \tag{2}$$

$$\begin{cases} u^{(4)}(x) = f(x), \quad x \in (a,b) \\ u(a) = u'(a) = u(b) = u'(b) = 0, \end{cases} \tag{3}$$

a) We start with writing the weak formulations of the three BVPs. Let $v \in V$ be a test function. We first do not specify the space V, and then verify what is the minimum regularity v has to have for the weak formulation to be well defined. We integrate all the equations on (a,b) against the test function v. We start with (1), where we remark that since the solution u has a known value at the extremum b, we can suppose $v(b) = 0$:

$$-\int_a^b u''(x)v(x)\mathrm{d}x + \int_a^b c(x)u(x)v(x)\mathrm{d}x = \int_a^b f(x)v(x), \tag{4}$$

which can be integrated by parts once so we can also impose the boundary conditions and get what follows

$$\begin{aligned} \int_a^b u'(x)v'(x)\mathrm{d}x - u'(x)v(x)|_a^b + \int_a^b c(x)u(x)v(x)\mathrm{d}x &= \int_a^b f(x)v(x)\mathrm{d}x \\ \int_a^b u'(x)v'(x)\mathrm{d}x + u'(a)v(a) + \int_a^b c(x)u(x)v(x)\mathrm{d}x &= \int_a^b f(x)v(x)\mathrm{d}x \\ \int_a^b u'(x)v'(x)\mathrm{d}x + (1 - \alpha u(a))v(a) + \int_a^b c(x)u(x)v(x)\mathrm{d}x &= \int_a^b f(x)v(x)\mathrm{d}x \\ \int_a^b u'(x)v'(x)\mathrm{d}x - \alpha u(a)v(a) + \int_a^b c(x)u(x)v(x)\mathrm{d}x &= \int_a^b f(x)v(x)\mathrm{d}x - v(a). \end{aligned}$$

This hence leads to the bilinear form

$$a(u,v) = \int_a^b u'(x)v'(x)\mathrm{d}x - \alpha u(a)v(a) + \int_a^b c(x)u(x)v(x)\mathrm{d}x$$

and the linear form

$$F(v) = \int_a^b f(x)v(x)\mathrm{d}x - v(a).$$

We notice that for the integrals to be well defined, one can take $v \in V := \{w \in H^1(a,b) : w(b) = 0\}$.
The weak formulation for (2) can be derived exactly in the same way. However, since u is known on both a and b, we can assume $v(a) = v(b) = 0$ and hence everything simplifies to

$$\int_a^b u'(x)v'(x)\mathrm{d}x + \int_a^b c(x)u(x)v(x)\mathrm{d}x = \int_a^b f(x)v(x)\mathrm{d}x.$$

As a cosequence, we can take $v \in V = H_0^1(a,b) = \{w \in H^1(a,b) : w(a) = w(b) = 0\}$ and set

$$a(u,v) = \int_a^b u'(x)v'(x)\mathrm{d}x + \int_a^b c(x)u(x)v(x)\mathrm{d}x, \quad F(v) = \int_a^b f(x)v(x)\mathrm{d}x.$$

We conclude with (3). The procedure is similar, but we have to integrate by parts twice since we have the 4th derivative appearing in the PDE. Because of the boundary conditions, we can suppose $v(a) = v(b) = v'(a) = v'(b) = 0$ for $v \in V$, and get

$$\begin{aligned}
\int_a^b u^{(4)}(x)v(x)\mathrm{d}x &= \int_a^b f(x)v(x)\mathrm{d}x \\
-\int_a^b u^{(3)}(x)v'(x)\mathrm{d}x + u^{(3)}(x)v(x)|_a^b &= \int_a^b f(x)v(x)\mathrm{d}x \\
\int_a^b u''(x)v''(x)\mathrm{d}x + u''(x)v'(x)|_a^b &= \int_a^b f(x)v(x)\mathrm{d}x \\
\int_a^b u''(x)v''(x)\mathrm{d}x &= \int_a^b f(x)v(x)\mathrm{d}x.
\end{aligned}$$

This allows to conclude that we can take $V = H_0^2(a,b) = \{w \in H^2(a,b) : w(a) = w(b) = w'(a) = w'(b)\}$ and set

$$a(u,v) = \int_a^b u''(x)v''(x)\mathrm{d}x, \quad F(v) = \int_a^b f(x)v(x)\mathrm{d}x.$$

b) We now suppose c is continuous and hence bounded. In this way the integrals involved in the bilinear forms of (2) and (1) are bounded. Indeed we have that the bilinear form for (1) satisfies

$$\begin{aligned}
a(u,v) &\leq \|u'\|_{L^2}\|v'\|_{L^2} + \|c\|_\infty\|u\|_{L^2}\|v\|_{L^2} + |\alpha||u(a)||v(a)| \\
&\leq (1 + \|c\|_\infty + |\alpha|)\|u\|_{H^1}\|v\|_{H^1}.
\end{aligned}$$

Alternatively, we can also bound this as

$$a(u,v) \leq (\max\{1, \|c\|_\infty\} + |\alpha|)\|u\|_{H^1}\|v\|_{H^1},$$

which is a tighter bound. We remark that the last term in the first line can be bounded with the H^1 norm since, for example, one can prove

$$|u(a)| = \left|\int_a^b u'(x)\mathrm{d}x\right| \leq \int_a^b |u'(x)|\mathrm{d}x \leq \sqrt{b-a}\|u'\|_{L^2} \leq \sqrt{b-a}\|u\|_{H^1}.$$

For (2) we instead have the same but the term with α does not appear. Thus both these bilinear forms are bounded, i.e. continuous. For the bilinear form of (3) we instead have

$$a(u,v) = \int_a^b u''(x)v''(x)\mathrm{d}x \leq \|u''\|_{L^2}\|v''\|_{L^2} \leq \|u\|_{H^2}\|v\|_{H^2}.$$

c) For coercivity related to the problem (1) we can assume $\alpha \leq 0$ and get

$$\begin{aligned} a(v,v) &= \int_a^b c(x)(v(x))^2\mathrm{d}x + \int_a^b (v'(x))^2\mathrm{d}x - \alpha v(a)^2 \\ &\geq \int_a^b c(x)(v(x))^2\mathrm{d}x + \int_a^b (v'(x))^2\mathrm{d}x \geq \min\left\{\min_{x\in[a,b]}|c(x)|, 1\right\}\|v\|_{H^1}^2. \end{aligned}$$

The reasoning for (2) is the same, where the term with α does not appear. For (3), we have

$$a(v,v) = \int_a^b (v''(x))^2\mathrm{d}x = |v|_{H^2}^2 \geq C_{a,b}\|v\|_{H^2}^2$$

for $C_{a,b}$ the constant given by Friedrich's inequality:

$$u \in W_0^{s,p}(\Omega) \implies \|u\|_{L^p(\Omega)} \leq C\left(\sum_{|\alpha|=s}\|\partial_\alpha u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}} = c|u|_{W^{s,p}(\Omega)}.$$

A simpler way to prove this last result is by applying Poincaré's inequality twice, since $v' \in H_0^1(a,b)$ too. Thus one gets

$$\begin{aligned} \|v'\|_{L^2} &\leq c\|v''\|_{L^2} \\ \|v\|_{L^2} &\leq c\|v'\|_{L^2} \leq c^2\|v''\|_{L^2} \\ \|v''\|_{L^2}^2 &\geq \tilde{c}\|v\|_{H^2}^2. \end{aligned}$$

d) For the boundedness of the linear functionals, we can assume $f \in L^2(a,b)$. This allows to conclude that

$$\int_a^b f(x)v(x)\mathrm{d}x \leq c\|v\|_V,$$

where $c = \|f\|_{L^2}$. The additional term appearing in (1) is also bounded by $\|v\|_V$. This allows to conclude the boundedness of the linear functionals F, and hence their continuity.

We thus remark that by Lax-Milgram's theorem, whenever the aforementioned assumptions on α, c, f hold true, there exist a unique solution to the weak problems associated to the three equations of interest.

2 This proof follows the reasoning presented in [1, Section 6.2.1]. For the sake of completeness of the solution, we report all the steps also here.

a) The existence of $w_u \in V$ such that $a(u,v) = \langle w_u, v\rangle$ is guaranteed by Riesz Representation theorem, as long as we prove $a(u,\cdot) : V \to \mathbb{R}$ is a linear and continuous functional. This comes immediatly by the boundedness of a, since

$$a(u,v) \leq C\|u\|\|v\|$$

and hence the boundedness constant of $a(u,\cdot)$ is $C\|u\|$.

b) By definition, we have

$$a(u, v) = \langle Au, v\rangle.$$

We also know that a is bilinear and continuous. Thus,

$$\begin{aligned}\langle A(\lambda u_1 + \gamma u_2), v\rangle &= a(\lambda u_1 + \gamma u_2, v) = \lambda a(u_1, v) + \gamma a(u_2, v)\\ &= \lambda\langle Au_1, v\rangle + \gamma\langle Au_2, v\rangle.\end{aligned}$$

This allows to prove the linearity of A, since v is an arbitrary element of V. For the boundedness, we notice that by setting $v = Au$

$$\langle Au, Au\rangle = \|Au\|^2 = a(u, Au) \leq C\|u\|\|Au\|,$$

and hence $\|Au\| \leq C\|u\|$.

c) Injectivity comes naturally from the coercivity of a. Indeed we have

$$\alpha\|u\|^2 \leq a(u, u) = \langle Au, u\rangle.$$

Suppose now $Au = 0$. Then $\langle Au, u\rangle = 0$, and hence one obtains

$$\alpha\|u\|^2 \leq 0,$$

which is possible only when $u = 0$ since $\alpha > 0$.

d) This point can be obtained in a similar way to the previous one. Indeed we know

$$\alpha\|u\|^2 \leq a(u, u) = \langle Au, u\rangle \leq \|Au\|\|u\|.$$

This implies $0 \leq \alpha\|u\| \leq \|Au\|$. Thus since Hilbert spaces are complete metric spaces, we have that a converging sequence is Cauchy. This means that for every $\varepsilon > 0$ there is $K_\varepsilon > 0$ such that for all $m, n > K_\varepsilon$

$$\|Au^m - Au^n\| \leq \varepsilon.$$

As a consequence, also

$$\|u^m - u^n\| \leq \varepsilon/\alpha,$$

and hence also $\{u^k\}$ is a Cauchy sequence. By completeness of V this allows to conclude that there is $u \in V$ such that $\|u^k - u\| \to 0$. We can conclude by continuity of A that $Au = v$.

e) Following the hint, we suppose $\text{range}(A) \subsetneq V$ and take $v \in \text{range}(A)^\perp \subset V$, $v \neq 0$. We remark that this can be done since $\text{range}(A)$ is closed in V. As a consequence we have

$$0 \leq \alpha\|v\| \leq a(v, v) = \langle Av, v\rangle = 0,$$

where the last equality comes from the fact that $Av \in \text{range}(A)$ and $v \in \text{range}(A)^\perp$. This is however a contradiction since it implies $v = 0$. It follows $\text{range}(A) = V$, and hence A is a bijective linear and continuous operator defined on V.

f) Since by assumption the linear functional $f(v) = \langle f, v\rangle$ is also bounded, i.e. continuous, we can apply again Riesz representation theorem. This guarantees the existence of an element $u_f \in V$ such that $f(v) = \langle u_f, v\rangle$ for every $v \in V$.

We are now in a setting that allows to conclude the proof since we have obtained

$$a(u, v) = \langle Au, v\rangle = \langle w_f, v\rangle = f(v)$$

for every v. Since A is a bijection on V, it follows that

$$Au = w_f$$

admits a unique solution. Furthermore, it is also possible to show that

$$\alpha\|u\| \leq \|Au\| = \|w_f\| = \|f\|,$$

and hence there is also a stability result of u that writes

$$\|u\| \leq \frac{\|f\|}{\alpha}.$$

3 We now consider the Poisson equation

$$\begin{cases} -u''(x) = 2, \quad x \in (0, 1) \\ u(0) = u(1) = 0. \end{cases} \tag{5}$$

a) To find the analytical solution of this problem one can follow various approaches due to the simplicity of the equation. We simply integrate the equation twice on both sides and impose the boundary conditions:

$$-\int u''(x)\mathrm{d}x = 2\int_0^1 \mathrm{d}x \implies -u'(x) + c_1 = 2x$$

and

$$-\int u'(x)\mathrm{d}x = \int (2x - c_1)\mathrm{d}x \implies u(x) = -x^2 + c_1 x - c_2.$$

Imposing the boundary conditions we get

$$u(0) = -c_2 = 0, \quad u(1) = c_1 - 1 = 0$$

and hence $u(x) = -x^2 + x$.

b) To write the weak formulation we follow the same approach as in exercise 1. We define $v \in H_0^1(0, 1)$ and integrate both sides of the equation against the test function v to get:

$$-\int_0^1 u''(x)v(x)\mathrm{d}x = 2\int_0^1 v(x)\mathrm{d}x$$
$$\int_0^1 u'(x)v'(x)\mathrm{d}x = 2\int_0^1 v(x)\mathrm{d}x$$

that gives

$$a(u, v) = \int_0^1 u'(x)v'(x)\mathrm{d}x, \quad F(v) = 2\int_0^1 v(x)\mathrm{d}x.$$

c) We now consider the finite dimensional space

$$V_h = \left\{ \varphi_k(x) := \frac{\sqrt{2}\sin((2k+1)\pi x)}{\pi(2k+1)} \,:\; k = 1, ..., n \right\}.$$

To prove that these basis functions are orthonormal with respect to $a(\cdot,\cdot)$ there are many options. To not give for granted trigonometrical identities, we rely only on the exponential representation of the $\cos(\cdot)$ function. Since the bilinear form a involves the first derivatives of the basis functions, we compute them

$$\varphi_k'(x) = \frac{\sqrt{2}(2k+1)\pi}{\pi(2k+1)}\cos((2k+1)\pi x) = \sqrt{2}\cos((2k+1)\pi x).$$

Recalling Euler's identity we can also write

$$\varphi_k'(x) = \frac{\sqrt{2}}{2}\left(e^{i(2k+1)\pi x} + e^{-i(2k+1)\pi x}\right).$$

Thus

$$(\varphi_k'(x))^2 = \frac{1}{2}\left(e^{2i(2k+1)\pi x} + e^{-2i(2k+1)\pi x} + 2\right) = \cos((2k+1)2\pi x) + 1.$$

Similarly, if $k \neq p$ we have

$$\begin{aligned}\varphi_k(x)\varphi_p(x) = \frac{1}{2}\Big(&e^{2i[(2k+1)+(2p+1)]\pi x} + e^{2i[(2k+1)-(2p+1)]\pi x} \\ &+e^{2i[-(2k+1)+(2p+1)]\pi x} + e^{-2i[(2k+1)+(2p+1)]\pi x}\Big).\end{aligned}$$

This can be simplified as

$$\varphi_k(x)\varphi_p(x) = \cos\left(2\pi x[2(k+p+1)]\right)) + \cos\left(2\pi x[2(k-p)]\right).$$

We recall that

$$\int_0^1 \cos\left(2r\pi x\right)\mathrm{d}x = 0, \quad r \in \mathbb{Z},$$

and this allows to conclude.

d) Let $u_h \in V_h$, i.e. $u_h = \sum_{k=1}^n u_k\varphi_k(x)$. Then the Ritz-Galerkin formulation of the problem writes : Find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$

Since v_h is arbitrary in V_h, this is equivalent to say that such relation has to hold for every $v_h \in \{\varphi_1, ..., \varphi_n\}$. By linearity and orthonormality of the basis functions the relation for $v_h = \varphi_p$ simplifies to

$$a(u_h, \varphi_p) = \sum_{k=1}^n u_k a(\varphi_k, \varphi_p) = u_p = f(\varphi_p) = 2\int_0^1 \varphi_p(x)\mathrm{d}x.$$

Which can be written extensively as: Find $u_h = \sum_{k=1}^n u_k\varphi_k(x)$ such that

$$u_k = 2\int_0^1 \varphi_k(x)\mathrm{d}x, \quad k = 1, ..., n. \tag{6}$$

e) The solution is very straightforward to write, since we only have to evaluate the right hand side in (6), i.e. to compute

$$\begin{aligned} u_k = 2\int_0^1 \varphi_k(x)\mathrm{d}x &= \frac{2\sqrt{2}}{\pi(2k+1)}\int_0^1 \sin\left((2k+1)\pi x\right)\mathrm{d}x \\ &= -\frac{2\sqrt{2}}{\pi^2(2k+1)^2}\cos\left((2k+1)\pi x\right)|_0^1 \\ &= -\frac{2\sqrt{2}}{\pi^2(2k+1)^2}(-1-1) \\ &= \frac{4\sqrt{2}}{\pi^2(2k+1)^2}. \end{aligned}$$

We write the solution in its complete form as

$$u_h(x) = \sum_{k=1}^{n} \frac{8}{\pi^3(2k+1)^3}\sin\left((2k+1)\pi x\right).$$

Remark : We notice that the space V_h is not really well suited for the approximation of the solution u. Indeed, as we increment k, the basis functions tend to shrink towards the zero function. The choice of this space is to simplify the calculations and get familiar with the Ritz-Galerkin formulation. A better approximation can be obtained by using

$$V_h' = \left\{\varphi_k(x) := \sin((2k+1)\pi x) : \ k = 1, ..., n\right\}.$$

In that case only orthogonality holds, and not orthonormality.

f) Galerkin orthogonality can be verified in multiple ways. We decide to do it by explicitly computing the needed terms. We first compute $u'(x) = -2x + 1$. We do not explicitly write the coefficients u_k we have computed, but we leave them as constants:

$$\begin{aligned} a(u-u_h, \varphi_p) &= \left\langle -2x + 1 - \sum_{k=1}^{n} u_k\varphi_k'(x), \varphi_p'(x)\right\rangle \\ &= -2\int_0^1 x\varphi_p'(x)\mathrm{d}x + \int_0^1 \varphi_p'(x)\mathrm{d}x - \sum_{k=1}^{n} u_k\int_0^1 \varphi_k'(x)\varphi_p'(x)\mathrm{d}x \\ &= -2x\varphi_p(x)|_0^1 + 2\int_0^1 \varphi_p(x)\mathrm{d}x - u_p \\ &= 2\int_0^1 \varphi_j(x)\mathrm{d}x - u_p \\ &= \frac{4\sqrt{2}}{\pi^2(2p+1)^2} - u_p = 0 \end{aligned}$$

since we know

$$u_p = \frac{4\sqrt{2}}{\pi^2(2p+1)^2}.$$

References

[1] Lawrence C Evans. *Partial differential equations*, volume 19. American Mathematical Society, 2022.