Norwegian University of Science and Technology
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TMA4220-2023 practice questions

You may use any results presented within a problem without proof, unless explicitly stated otherwise.

Problem 1 Let $u \in H^2(\Omega)$ solve the Poisson equation

$$\Delta u = f \text{ in } \Omega \subset \mathbb{R}^3$$
$$\nabla u \cdot \vec{n} = 0 \text{ on } \partial\Omega$$

for some data $f \in L_2(\Omega)$ where $\vec{n}$ is the unit normal vector to $\partial\Omega$.

a) Define the function spaces $H^2(\Omega)$ and $L_2(\Omega)$.

b) Show that the Poisson equation can be written as seeking $u \in \mathbb{V}$ such that

$$a(u, v) = g(v) \qquad \forall v \in \mathbb{V}, \tag{1}$$

by finding the bilinear form $a(\cdot, \cdot)$ and bounded linear functional $g(v)$.

c) What is the natural choice of $\mathbb{V}$? Justify this choice.

For the remainder of this question you may assume the following results without prove.
Poincaré inequality: Suppose $\Omega \in \mathbb{R}^n$ is bounded and $u \in \mathbb{V}$, then there exists a constant C such that

$$\|u\|_0 \leq C_\Omega \|\nabla u\|_0.$$

Riesz representation theorem: Let $\mathbb{V}$ be a Hilbert space and $g \in \mathbb{V}^*$ be a bounded linear functional on the dual space of $\mathbb{V}$. Then, there exists a unique $u \in \mathbb{V}$ such that

$$g(v) = \langle u, v \rangle_{\mathbb{V}} \qquad \forall v \in \mathbb{V}.$$

d) Prove that the bilinear form you found in part (b) is elliptic in $\mathbb{V}$ and symmetric. If you could not complete part (b) choose any non-degenerate bilinear form. If you could not complete part (c) choose any Hilbert space.

e) Prove for any symmetric and $\mathbb{V}$-elliptic bilinear form $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ there is a unique solution to (1). Is the weak form you found in (b) well posed?

Problem 2

a) Give Ciarlet's definition of a finite element. Remark on one thing which is missing from this definition which is required to implement a finite element.

b) Define the quadratic Lagrange finite element and the cubic Hermite elements over triangular meshes. This may be done through sketching the element over an arbitrary triangle and describing the sketch.

c) What is a hyperplane? You may either give the definition or a convincing heuristic explanation.

Throughout the remainder of this question you will require the following.
Definition: $\mathcal{N}$ determines $\mathcal{P}$ if for $\psi \in \mathcal{P}$ with $N(\psi) = 0 \ \forall N \in \mathcal{N}$.

Deconstruction lemma: Let P be a polynomial of degree $d \geq 1$ vanishing on the hyperplane L. Then we have $P = LQ$, where Q is polynomial of degree $d - 1$.

d) Prove the quadratic Lagrange element satisfies Ciarlet's definition of a finite element.

e) Prove the cubic Hermite element satisfies Ciarlet's definition of a finite element.

Problem 3 In this question we focus on the weak formulation of seeking $u \in H_0^1(\Omega)$ such that

$$\langle \nabla u, \nabla v \rangle + c \langle u, v \rangle = \langle f, v \rangle \qquad \forall v \in H_0^1(\Omega). \tag{2}$$

a) Assuming $u \in H_0^1 \cap H^2$, find the strong form of (2). Infer the pointwise form of equation and associated boundary conditions.

Lax-Milgram theorem: Let $\mathbb{V}$ be a Hilbert space, $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be $\mathbb{V}$-elliptic, and $g : \mathbb{V} \to \mathbb{R}$ be a bounded linear functional on $\mathbb{V}$. Then, there exists a unique $u \in \mathbb{V}$ such that $a(u, v) = g(v)$ for all $v \in \mathbb{V}$.

b) What does it mean for a bilinear form to be $\mathbb{V}$-elliptic?

c) Formulate (2) over a finite dimensional subspace $\mathbb{V}_h \subset \mathbb{V}$ and prove the discretisation has a unique solution.

Cea's lemma: If $u_h \in \mathbb{V}_h$ solves the finite element approximation found in (c) and $u \in \mathbb{V}$ solves the weak form (2), then

$$\|u - u_h\|_{\mathbb{V}} \leq C \min_{v_h \in \mathbb{V}_h} \|u - v_h\|_{\mathbb{V}}.$$

d) Prove Cea's lemma and determine the constant C.

Accuracy of Lagrange finite elements: Let $\{\mathcal{T}_h\}$ for $0 < h \leq 1$ be a nondegenerate family of polyhedral triangulations for $\Omega \in \mathbb{R}^n$, and let $\mathbb{V}_h$ be the space of degree k finite elements where $k + 1 - \frac{n}{2} > 0$ and $\mathcal{I} : C^0(\Omega) \to \mathbb{V}_h$ is the interpolant. Then, there exists a constant C, independent of h, such that if $u \in H^{k+1}$ then

$$\left(\sum_{T \in \mathcal{T}_h} \|u - \mathcal{I}u\|_s^2 \right)^{\frac{1}{2}} \leq C h^{k+1-s} |u|_{k+1}$$

for $0 \leq s \leq k+1$.

e) If $\mathbb{V}_h$ is spanned by order k Lagrange finite elements over a family of meshes $\{\mathcal{T}_h\}$ satisfying the above theorem, prove the order of convergence of the approximation found in part (c) when u is infinitely smooth (i.e., $u \in C^\infty$).

f) If $u \in H^3$, what is the optimal polynomial degree $\hat{k}$ to chose for Lagrange finite elements? Justify this choice.

Problem 4

Let $u \in H^1([0,T], H_0^1(\Omega))$ describe the weak solution of the heat equation, i.e.,

$$\begin{aligned} \langle u_t, v \rangle + \langle \nabla u, \nabla v \rangle &= \langle f, v \rangle \qquad && \forall v \in H_0^1(\Omega) \\ u(0,x) &= u_0(x) && \text{for } u_0(x) \in H_0^1(\Omega)\text{'} \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the $L_2(\Omega)$ inner product over the spatial domain.
Poincaré inequality: Suppose $\Omega \in \mathbb{R}^n$ is bounded and $u \in \mathbb{V}$, then there exists a constant C such that

$$\|u\|_0 \leq C_\Omega \|\nabla u\|_0.$$

a) Show, for the continuous weak form, that

$$\|u(t)\| \leq \|u_0\| + C \int_0^t \|f(s)\| \mathrm{d}s,$$

and determine the constant C. *Hint: start by defining the bilinear form $a(\cdot,\cdot)$ and show that it is coercive.*

Now we restrict to the finite dimensional setting in space, i.e., seek $u_h \in C^1([0,T], \mathbb{V}_h)$ such that

$$\begin{aligned} \langle u_{h,t}, v_h \rangle + \langle \nabla u_h, \nabla v_h \rangle &= \langle f, v_h \rangle \qquad \forall v_h \in \mathbb{V}_h(\Omega) \\ u_h(0,x) &= \Pi_{\mathbb{V}_h} u_0(x), \end{aligned}$$

where $\Pi_{\mathbb{V}_h}$ is the L_2 projection into the space $\mathbb{V}_h$. Note we restrict $\mathbb{V}_h$ such that $\mathbb{V}_h \subset \mathbb{V}$ and $\mathbb{V}_h = \text{span}(\{\phi_i\})$.

b) Write the underlying system in terms of a mass matrix M, stiffness matrix M, right hand side b and initial data. Write an arbitrary element of each of these matrices and vectors in terms of the basis functions.

c) Prove that M is positive definite.

d) Write the fully discrete method if discretising in time with backward Euler.

e) Find an explicit expression for the solution u_h in terms of the degrees of freedom of the underlying system. You may use that A is positive semi-definite without proof. Relate this expression back to the finite element function u_h.

f) Let $f = 0$, then prove for any positive time step that the fully discrete method found in (e) is stable over time. You may argue using the underlying linear system or functional analysis arguments.

Problem 5 In this question we consider the spatially discrete, temporally continuous solution of the heat equation. To be more concise, seek $u_h \in C^1([0,T], \mathbb{V}_h)$ such that

$$\begin{aligned} \langle u_{h,t}, v_h \rangle + \langle \nabla u_h, \nabla v_h \rangle &= \langle f, v_h \rangle \qquad \forall v_h \in \mathbb{V}_h(\Omega) \\ u_h(0,x) &= \Pi_{\mathbb{V}_h} u_0(x), \end{aligned} \tag{3}$$

where $\Pi_{\mathbb{V}_h} : H^1(\Omega) \to \mathbb{V}_h$ is the L_2 projection and $\mathbb{V}_h \subset H_0^1(\Omega)$. Here we shall study the convergence of the spatially discrete problem, which requires the following:
Ritz projection: The Ritz (or elliptic) projection is defined by $\mathcal{R}_h : H_0^1 \to \mathbb{V}_h$ such that for any $v \in \mathbb{V}_h$

$$a(\mathcal{R}_h v - v, \chi) = 0 \qquad \forall \chi \in \mathbb{V}_h, \tag{4}$$

where $a(\cdot,\cdot)$ is the bilinear form induced by the elliptic contribution of (3).
Regularity assumption on initial data: Let $\Omega \subset \mathbb{R}^d$ and assume we have a family of regular triangulations $\{\mathcal{T}_h\}$ with $\mathcal{T}_h \subset H_0^1(\Omega)$ such that, for some $r \geq 2$ and small enough h

$$\inf_{\chi \in \mathbb{V}_h} \{\|u_0 - \chi\| + h\|\nabla(u_0 - \chi)\|\} \leq C h^s \|u_0\|_s \text{ for } 1 \leq s \leq r,$$

when $u_0 \in H^s \cap H_0^1$ with u_0 as described in (3).

a) Prove that the Ritz projection (4) is stable, i.e., that

$$|\mathcal{R}_h v|_1 \leq |v|_1,$$

where $|\cdot|_1$ is the H^1 semi-norm.

b) Prove that, when applied to smooth enough functions, the Ritz projection converges optimally in the H^1 semi-norm.

For the remainder of this question you may use that the Ritz projection converges optimally in L_2 (you do not need to prove optimal convergence through an elliptic regularity argument).

c) Through splitting the error into elliptic and parabolic parts through defining

$$u_h(t) - u(t) = u_h(t) - \mathcal{R}_h u(t) + \mathcal{R}_h u(t) - u(t) =: \theta(t) + \rho(t),$$

prove *one* of the following two results:

1. Let $u \in C^1([0,T], H^s(\Omega))$ for $s \leq r$ solve the heat equation and u_h be the finite dimensional approximation given in (3). Further assume that the regularity assumption on the initial data is satisfied, then

$$\|u_h(t) - u(t)\| \leq \|\Pi_{\mathbb{V}_h} u_0 - u_0\| + Ch^r \left(\|u_0\|_s + \int_0^t \|u_\tau\|_s \mathrm{d}\tau \right).$$

2. Let $u \in C^1([0,T], H^s(\Omega))$ for $s \leq r$ solve the heat equation and u_h be the finite dimensional approximation given in (3). Further assume that the regularity assumption on the initial data is satisfied, then

$$\begin{aligned}\|\nabla(u_h(t) - u(t))\|^2 \leq C\|\nabla(\Pi_{\mathbb{V}_h} u_0 - u_0)\|^2 \\ + Ch^{2s-2} \left(\|u_0\|_s^2 + \|u\|_s^2 + \int_0^t \|u_\tau\|_{s-1}^2 \mathrm{d}\tau \right).\end{aligned}$$

Problem 6 Let $u \in H^2(\Omega)$ solve

$$\begin{aligned}\Delta u &= f \text{ in } \Omega \subset \mathbb{R}^3 \\ u &= 0 \text{ on } \partial\Omega,\end{aligned} \tag{5}$$

where $f \in H^{-1}(\Omega)$.

a) Through the auxiliary variable $\sigma = \nabla u$ rewrite (5) in the form

$$\begin{aligned} a(\sigma, w) + b(w, u) &= 0 \qquad \forall w \in \mathbb{W} \\ b(\sigma, v) &= F(v) \qquad \forall v \in \mathbb{V}. \end{aligned} \tag{6}$$

Determine the bilinear forms a and b, and the bounded linear functional F.

b) What does it mean for the bilinear forms $a : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$ and $b : \mathbb{W} \times \mathbb{V}$ to be elliptic over $\mathbb{W} \times \mathbb{V}$?

Well-posedness theorem: Let $\mathbb{W}$, $\mathbb{V}$ be Hilbert spaces. Suppose $a : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$ and $b : \mathbb{W} \times \mathbb{V}$ are bounded bilinear functionals and F is a bounded linear functional on $\mathbb{V}$. If a is coercive on $\tilde{\mathbb{W}} := \{w \in \mathbb{W} \mid b(w, v) = 0 \ \forall v \in \mathbb{V}\}$ and b satisfies an inf-sup condition there exists a unique solution $(\sigma, u) \in (\mathbb{W}, \mathbb{V})$ solving (6).

c) If $\mathbb{W} = H(\mathrm{div})$ and $\mathbb{V} = L_2$ prove the weak formulation found in (b) has a unique solution.

Problem 7

In this question we consider the nonconforming approximation of the Poisson equation with zero Dirichlet boundary conditions described by seeking $u_h \in \mathbb{V}_h(\Omega)$ such that

$$a_h(u_h, v_h) = \langle f, v_h \rangle \qquad \forall v_h \in \mathbb{V}_h(\Omega), \tag{7}$$

where $f \in L_2(\Omega)$ and $\mathbb{V}_h \not\subseteq H_0^1(\Omega)$. In particular, we choose

$$\begin{aligned} a_h(u_h, v_h) = &\sum_{T \in \mathcal{P}_h} \int_T \nabla u_h \cdot \nabla v_h \mathrm{d}x \\ &- \sum_{e \in \mathcal{E}_h} \int_e \Big(\{\nabla u_h\} \cdot [\![v_h]\!] + \{\nabla v_h\} \cdot [\![u_h]\!] \Big) \mathrm{d}s + \sum_{e \in \mathcal{E}_h} \frac{\eta}{|e|} \int_e [\![u]\!] \cdot [\![v]\!] \mathrm{d}s, \end{aligned} \tag{8}$$

where η is sufficiently big, $\mathcal{P}_h$ is the set of elements in the mesh and $\mathcal{E}_h$ the set of edges over the mesh.

a) The three flux terms are individually referred to as the consistency term, symmetry term and penalty term. Identify which is which and heustically justify the naming convention.

b) For a function $v_h \in \mathbb{V}_h(\Omega)$ with $\Omega \subset \mathbb{R}^d$, give the definition of the jump $[\![v_h]\!]$ and average $\{v_h\}$.

c) Prove that the interior penalty bilinear form is consistent. That is to say, apply a function $u \in H^2 \cap H_0^1(\Omega)$ to (7) and recover strong form of the Poisson equation tested with $v_h \in \mathbb{V}_h$.

d) Which mesh dependent norm is the interior penalty bilinear form (8) elliptic with respect to?

Lemma: There exists a constant $C > 0$ depending on mesh regularity such that

$$\sum_{e \in \mathcal{E}_h} |e| \|\{\nabla v_h\}\|_{L_2(e)}^2 \leq C \sum_{T \in \mathcal{P}_h} \|\nabla v_h\|_{L_2(T)}^2.$$

e) Prove that $a_h(\cdot,\cdot)$ is continuous in the norm chosen in (d).

f) Prove that $a_h(\cdot,\cdot)$ is coercive in the norm chosen in (d).