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TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Fall 2023

Exercise set 3

1 Let $\Omega \subset \mathbb{R}^n$ be a compact set with smooth boundary $\partial\Omega$. We consider the heat equation

$$\begin{cases} \partial_t u - \Delta u = f, \ \Omega, \ t \geq 0 \\ u(t,x) = 0, \ x \in \partial\Omega \\ u(0,x) = v \in L^2(\Omega), \ x \in \Omega. \end{cases} \tag{1}$$

a) Write the continuous weak formulation, specifying also the proper functional spaces where it is defined.

b) Let us now consider semidiscretised problem obtained by applying the finite element discretisation in space to (1) and defining it on the discrete space

$$\mathbb{V}_h = \text{span}\{\Phi_1, ..., \Phi_N\}.$$

Here with h we denote the spatial mesh size. Write the discrete weak problem in vector-matrix notation and prove that the stiffness matrix is symmetric and positive semidefinite.

c) Prove there is a unique solution to the semi-discretized problem derived in the previous question.

d) We now analyse the convergence properties of the *Backward-Euler Galerkin method* applied to approximate the solution to the time dependent semi-discrete problem. This method is defined by approximating the time derivative $\partial_t u(t_n)$ by the backward difference

$$\partial_t u(t_n) \approx \bar{\partial} U^n := \frac{U^n - U^{n-1}}{\delta t}$$

where $t_n = n\delta t$ and δt is the selected time step. Let now u be the solution of (1). Derive an estimate of the form

$$\|\nabla(U^n - u(t_n))\| \leq C(u)(h^{r-1} + \delta t)$$

writing explicitly the function $C(u)$. You can suppose there is a $v_h \in S_h$ such that

$$\|v_h - v\| \leq Ch^r \|v\|_r$$

and $v(x,t) = 0$ for $x \in \partial\Omega$.

2 We now consider the Stokes problem and aim to prove the well posedness of the weak formulation. Let $\Omega \subset \mathbb{R}^d$ be a smooth compact set, $f \in L^2(\Omega)$ and consider the problem

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = f & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$. We remark that

$$\mathbb{R}^d \ni \Delta \boldsymbol{u},\ (\Delta \boldsymbol{u})_i = \Delta(\boldsymbol{u}_i).$$

a) We now want to find the functional spaces W, V, the bilinear forms $a : W \times W \to \mathbb{R}$ and $b : W \times V \to \mathbb{R}$, and the linear functional $F : W \to \mathbb{R}$ defining the weak formulation of (2) as

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= F(\boldsymbol{v}),\ \forall \boldsymbol{v} \in W \\ b(\boldsymbol{u}, q) &= 0,\ \forall q \in V. \end{aligned} \tag{3}$$

Integrate (2) against the test functions $\boldsymbol{v} \in W$ and $q \in V$, and write the explicit expressions of the terms in (3).

b) Prove that $a : W \times W \to \mathbb{R}$ and $b : W \times V \to \mathbb{R}$ are continuous bilinear forms.

c) Prove that $a : W \times W \to \mathbb{R}$ is coercive.

d) Let us now consider the test function $q = 1 \in L^2(\Omega)$. What can we say about the value $b(\boldsymbol{u}, q)$? Motivate why it is not enough to set $V = L^2(\Omega)$ to define a well posed problem, and why

$$V = L^2_0(\Omega) := \left\{ p \in L^2(\Omega) : \int_\Omega p(x)dx = 0 \right\}$$

is a better choice.

e) Consider the Poisson equation

$$\begin{cases} -\Delta \phi = q,\ \Omega \\ \nabla \phi \cdot n = 0,\ \partial\Omega \end{cases}$$

for $q \in V$. Show this problem is well posed on $H^2(\Omega) \cap L^2_0(\Omega)$, and then set $\boldsymbol{u} := \nabla \phi$. By smoothness assumption of Ω, it is possible to get $\phi \in H^2(\Omega)$ and the regularity bound

$$\|\phi\|_2^2 \leq \beta \|q\|_V^2.$$

An immediate consequence is that

$$\boldsymbol{u} \cdot n = \nabla \phi \cdot n = 0,\ \Omega.$$

With a technical argument, one can also prove that $\boldsymbol{u} = 0$ on $\partial\Omega$ and hence $\boldsymbol{u} \in H^1_0(\Omega)$.

Use these results to prove that the inf-sup condition is satisfied by $b(\cdot, \cdot)$.

f) Is the problem defined in (3) well posed?

3 This exercise focuses on nonconforming finite element methods. Throughout this exercise we denote $\mathbb{V}_h$ as a discontinuous finite element space, and crucially note that $\mathbb{V}_h \nsubseteq H^1$. In particular we focus on the interior penalty method applied to solve the Poisson equation on a smooth domain Ω:

$$\begin{cases} -\Delta u = f, \ \Omega \\ u = g, \ \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. Instead of considering the symmetric interior penalty method, we consider the non-symmetric one, i.e. the one defined by the bilinear form

$$\begin{aligned} a_h^+(w,v) = &\sum_{T\in\mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla w\} \cdot [v] \, ds \\ &+ \sum_{e\in\mathcal{E}_h} \int_e \{\nabla v\} \cdot [w] \, ds + \frac{\eta}{|e|} \sum_{e\in\mathcal{E}_h} \int_e [w] \cdot [v] \, ds, \end{aligned} \tag{4}$$

where $\mathcal{T}_h = \{T\}$ is a triangulation of the domain Ω, and $\mathcal{E}_h$ is the collection of the edges of such triangles. The interior penalty method we consider is then defined as: Find $u_h^+ \in \mathbb{V}_h$ such that

$$a_h^+(u_h^+, v) = \int_\Omega fv \, dx, \ \forall v \in \mathbb{V}_h, \tag{5}$$

where $\mathbb{V}_h = \{v \in L^2(\Omega) : \ v_T = v|_T \in \mathcal{P}_r(T), \ \forall T \in \mathcal{T}_h\}$.

Recall the mesh-dependent energy norm

$$\|v\|_{\text{ip}}{}^2 = \sum_{T\in\mathcal{T}_h} \|\nabla v\|^2_{L^2(T)} + \eta^{-1} \sum_{e\in\mathcal{E}_h} |e| \|\{\nabla v\}\|^2_{L^2(e)} + 2\eta \sum_{e\in\mathcal{E}_h} |e|^{-1} \|[v]\|^2_{L^2(e)}. \tag{6}$$

a) Prove that a^+ defined in (4) is continuous in the norm (6).

b) Prove that a^+ defined in (4) is coercive in the modified mesh-dependent energy norm

$$\|v\|^2_{\text{dG}} = \sum_{T\in\mathcal{T}_h} \|\nabla v\|^2_{L^2(T)} + \eta \sum_{e\in\mathcal{E}_h} |e|^{-1} \|[v]\|^2_{L^2(e)}.$$

c) Prove that

$$\|v\|_{\text{ip}} \leq C(1+\eta^{-1}) \|v\|_{\text{dG}} \quad \forall v \in \mathbb{V}_h$$

for some positive constant C.

d) Prove Cèa's lemma for the weak formulation (5) by combining the results just derived. More explicitly, prove that there is a constant $C > 0$ such that

$$\|u - u_h^+\|_{\text{ip}} \leq C(1+\eta^{-1})^2 \inf_{v\in\mathbb{V}_h} \|u - v\|_{\text{ip}}.$$