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Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Fall 2023

Exercise set 2

1 Consider a bounded set $\Omega \subset \mathbb{R}^n$ with $\mathcal{C}^2$ boundary $\partial\Omega$. We define the following boundary value problem:

$$\begin{cases} -\Delta u + cu = f, & \Omega \\ u = 0, & \partial\Omega, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$, and $c > 0$ is a real number. We suppose $s \geq 2$ and $u \in H^s(\Omega) \cap H_0^1(\Omega)$, which is more regularity than what is needed for the existence and uniqueness of the weak solution. This assumption guarantees that

$$\|u\|_{H^2(\Omega)} < k\|f\|_{L^2(\Omega)}, \tag{2}$$

for a positive constant k. Furthermore, assume that on Ω is defined a family of regular triangulations $\{\mathcal{T}_h\}$ such that, for some $r \geq 2$, and small h,

$$\inf_{v_h \in V_h} \left\{ \|u - v_h\|_{L^2(\Omega)} + h\|\nabla u - \nabla v_h\|_{L^2(\Omega)} \right\} \leq \bar{c}h^s \|u\|_{H^s(\Omega)}$$

for $1 \leq s \leq r$ when $u \in H^s(\Omega) \cap H_0^1(\Omega)$.

a) Prove that called $u_h \in V_h \subset H_0^1(\Omega)$ the finite element solution to (1), it holds

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^{s-1}\|u\|_{H^s(\Omega)},$$

for a $C > 0$.

b) Prove that called $u_h \in V_h \subset H_0^1(\Omega)$ the finite element solution to (1), it holds

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^s\|u\|_{H^s(\Omega)},$$

for a $C > 0$. **Hint:** Use elliptic regularity assumption in (2) and define some appropriate dual problem of the form $-\Delta\psi + c\psi = \varphi$.

2 This exercise aims to give an understanding of what barycentric coordinates are, and how they are useful in the assembly of the stiffness matrix, mass matrix, and load vector of a finite element solver.

Let $T \subset \mathbb{R}^2$ be a triangle of vertices $p_1, p_2, p_3 \in \mathbb{R}^2$. The barycentric coordinates in the plane $\mathbb{R}^2$ are a triple (ξ_1, ξ_2, ξ_3) satisfying $\xi_1 + \xi_2 + \xi_3 = 1$. They can take arbitrary values on the real line. Let $p \in \mathbb{R}^2$ be a point of the plane. Its barycentric coordinates are defined as

$$p = \xi_1 p_1 + \xi_2 p_2 + \xi_3 p_3.$$

a) Motivate why if p is an interior point of the triangle T, then $\xi_1, \xi_2, \xi_3 > 0$.

b) Show that if p is on the boundary ∂T, then at least one of the coordinates is 0.

c) Show that if we set $\xi_3 = 1 - \xi_1 - \xi_2$, then the pair (ξ_1, ξ_2) expresses the coordinates of p in the reference frame with origin in p_3, and basis vectors $p_2 - p_3$, $p_1 - p_3$.

d) The Gaussian quadrature rule we have applied in 1d for the project, extends to the 2d case. In particular, one has that on the reference triangle T_r with vertices $(0,0), (1,0), (0,1)$, the integral

$$\int\int_{T_r} f(x,y)\mathrm{d}x\mathrm{d}y$$

can be approximated as $\sum_{i=1}^{p} w_i f(x_i, y_i)$, where $w_1, \ldots, w_p$ are the weights of the quadrature rule, and $\{(x_i, y_i)\}_{i=1}^{p}$ are the nodes of the quadrature rule. Let T be a generic triangle of vertices p_1, p_2, p_3. Write the change of variable that allows to pass from T_r to T and hence apply such a quadrature rule to approximate the integral

$$\int\int_{T} f(x,y)\mathrm{d}x\mathrm{d}y.$$

Hint: Think to how barycentric coordinates can help in designing the change of variable.

e) Let $\Omega \subset \mathbb{R}^2$ be a bounded set, and $\mathcal{T} = \{T_j\}_{j=1}^{n}$ be a regular triangulation of Ω. Let $v_1, \ldots, v_N \in \mathbb{R}^2$ be the nodes of the triangles defining the mesh. Recall that to assemble the stiffness matrix A, the mass matrix M, and the load vector F of a boundary value problem, one has to solve integrals over the triangles. We now workout a relatively simple example. The three point Gaussian Quadrature formula writes

$$\int\int_{T_r} f(\xi_1, \xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 = \frac{f(1/2,1/2) + f(1/2,0) + f(0,1/2)}{6}. \tag{3}$$

On the triangle T of vertices p_1, p_2, p_3, we consider the basis functions $\varphi_1, \varphi_2, \varphi_3 : T \to \mathbb{R}$ such that $\varphi_i(p_j) = 1$ if $i = j$ and 0 otherwise. When $p_1 = (0,0)$, $p_2 = (1,0)$, $p_3 = (0,1)$, one has

$$\begin{aligned} \varphi_1(\xi_1, \xi_2) &= 1 - \xi_1 - \xi_2, \\ \varphi_2(\xi_1, \xi_2) &= \xi_1, \\ \varphi_3(\xi_1, \xi_2) &= \xi_2. \end{aligned} \tag{4}$$

Write the basis functions for a generic triangle T of vertices $p_1 = (p_1^x, p_1^y), p_2 = (p_2^x, p_2^y), p_3 = (p_3^x, p_3^y)$. Then, use the quadrature rule in (3) to approximate the integrals

$$\int\int_{T} \varphi_1(x,y)\varphi_2(x,y)\mathrm{d}x\mathrm{d}y,$$

$$\int\int_{T} \varphi_1(x,y)\sin\left(2\pi(x+y)\right)\mathrm{d}x\mathrm{d}y,$$

where $p_1 = (0,0)$, $p_2 = (1,1)$, $p_3 = (0,1)$. In a generic finite element code, these integrals would correspond respectively to the entry $1, 2$ in the local mass

matrix, and the first entry of the local load vector, where the source term is $f(x, y) = \sin(2\pi(x + y))$.

Hint: Recall that using the change of variable in the previous point, you get the expressions in (4) for the basis functions.

3 **a)** Consider the quartic Lagrange finite element with nodes depicted in Figure 1. Let $\mathcal{P}^4(T)$ be the set of polynomials of degree 4 on the triangle T described in the figure. Consider the set of functions

$$\mathcal{N}_4 = \{N_1, \ldots, N_{15}\},$$

and call $\mathcal{V} = \{x_1, \ldots, x_{15}\}$ the nodes defined on T. Define the degrees of freedom N_i by their action over a generic function $u : T \to \mathbb{R}$, which is

$$N_i(u) := u(x_i).$$

Prove that $(T, \mathcal{P}^4(T), \mathcal{N})$ is a well defined finite element.

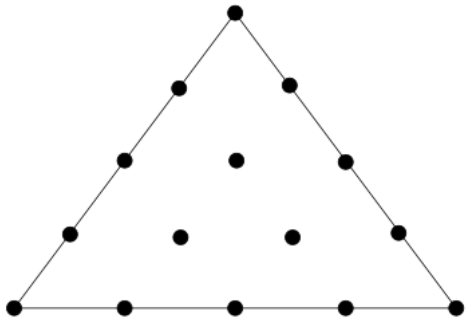

Figure 1: Lagrange element of Degree 4 over a triangle T

b) The quintic Argyris element is represented in Figure 2. Let $\mathcal{P}^5(T)$ be the polynomials of degree 5 on T, and consider the set of functions $\mathcal{N} = \{N_1, \ldots, N_{21} : \mathcal{P}^5(T) \to \mathbb{R}\}$. The way in which the N_is act depends on the i, and they are explicited in the figure. More precisely, we denote with $\bullet$ the points where the corresponding N_i evaluates on the point, with the inner circle we denote the evaluation of the gradient at the center of the circle, and with the outer circle we denote the evaluation of the three second derivatives at the center of the circle. The arrows represent the evaluation of the normal derivatives at the midpoints of the edges. Prove that $(T, \mathcal{P}^5(T), \mathcal{N})$ is a well defined finite element, i.e. that $\mathcal{N}$ determines $\mathcal{P}^5(T)$.

4 Consider a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary Ω. Let $f \in L^2(\Omega)$, and $g \in L^2(\partial\Omega)$, so that the boundary value problem

$$\begin{cases} -\Delta u + u = f, & \Omega \\ \frac{\partial u}{\partial n} = g, & \partial\Omega \end{cases} \tag{5}$$

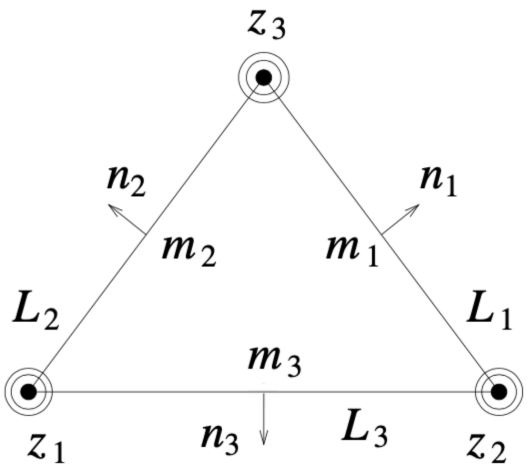

Figure 2: Quintic Argyris element

admits a unique solution for smooth enough u. We define

$$a(u,v) := \int_\Omega \left[\nabla u(x)\cdot\nabla v(x) + u(x)v(x)\right]\mathrm{d}x,$$
$$\ell(v) := \int_\Omega f(x)v(x)\mathrm{d}x + \int_{\partial\Omega} g(x)v(x)\mathrm{d}s.$$

The boundary value problem (5) admits a unique weak solution $u \in H^1(\Omega)$, which is a function such that $a(u,v) = \ell(v)$ for every $v \in H^1(\Omega)$. Consider then a finite dimensional subspace $V_h \subset H^1(\Omega)$ where we can find the solution to the Ritz-Galerkin formulation $u_h \in V_h$. This function satisfies $a(u_h, v_h) = \ell(v_h)$ for every $v_h \in V_h$. Let $\mathcal{T} = \{K_i\}_{i=1}^N$ be a regular triangulation of Ω.

Define the error $e(x) = u(x) - u_h(x)$.

a) Write explicitly the expression for the residual equation $a(e,v)$ for a generic $v \in H^1(\Omega)$.

b) Write explicitly the expression of $a(e,v_h)$ for a generic $v_h \in V_h$.

c) Write $a(e,v)$ as the sum of contributions over the single elements, i.e. find $a_i(e,v)$ so that

$$\begin{aligned} a(e,v) &= \sum_{i=1}^N \left(\int_{K_i} r(x)v(x)\mathrm{d}x + \int_{\partial K_i\cap\partial\Omega} R(x)v(x)\mathrm{d}s - \int_{\partial K_i\setminus\partial\Omega} \frac{\partial u(x)}{\partial n} v(x)\mathrm{d}s \right) \\ &=: \sum_{i=1}^N a_i(e,v), \end{aligned}$$

where $\partial K_i \cap \partial\Omega$ is the set of edges of K_i that are also on the boundary of Ω, while $\partial K_i \setminus \partial\Omega$ are the other ones.

d) Consider the interpolation operator $I_{V_h} : H^1(\Omega) \to V_h$ defined as in the lecture notes. Following the proof done in class, plug in $I_{V_h}u$ in the quantities we just

derived, and prove that there is a constant $C > 0$ such that

$$\|e\|^2_{H^1(\Omega)} \leq C \sum_{i=1}^{N} \left(h^2_{K_i} \|r\|^2_{L^2(K_i)} + h_{K_i} \|R\|^2_{L^2(\partial K_i)} \right).$$

e) What is the expression of the local estimator η_{K_i} on the element $K_i \in \mathcal{T}$?

f) Propose an experimental methodology for verifying (not proving) if this estimator is sharp and reliable.