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TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Fall 2023

Exercise set 1

For all the exercise sheet, we denote with V a Hilbert space with inner product

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

that induces the norm $\| \cdot \| : V \to \mathbb{R}$ defined as $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$ for all $v \in V$.

1 Consider the three following boundary value problems (BVPs) on $\Omega = [a, b] \subset \mathbb{R}$:

$$\begin{cases} -u''(x) + c(x)u(x) = f(x), \ x \in (a, b) \\ u'(a) + \alpha u(a) = 1, \ \alpha \in \mathbb{R} \\ u(b) = 0, \end{cases} \tag{1}$$

$$\begin{cases} -u''(x) + c(x)u(x) = f(x), \quad x \in (a, b) \\ u(a) = u(b) = 0, \end{cases} \tag{2}$$

$$\begin{cases} u^{(4)}(x) = f(x), \quad x \in (a, b) \\ u(a) = u'(a) = u(b) = u'(b) = 0, \end{cases} \tag{3}$$

where $u^{(4)}(x) = \frac{d^4}{dx^4} u(x)$.

a) Write the weak formulation of each of the three BVPs, specifying the functional space V to which the solution and the test functions belong.

b) Are the three bilinear forms continuous (i.e. bounded)? Specify what assumptions on c are necessary for this to hold.

c) Are the three bilinear forms coercive? Specify what assumptions on c are necessary for this to hold.

d) Are the three linear forms continuous (i.e. bounded)? Specify what assumptions on f are necessary for this to hold.

2 This exercise aims to build a proof for the Lax-Milgram theorem. The steps are guided by the points of the exercise. We first report the statement of the theorem, and also of Riesz representation theorem that helps in writing the proof.

Theorem 1 (Lax-Milgram)**.** *Let*

$$a : V \times V \to \mathbb{R}$$

be a bilinear form, for which there exist constants $\alpha, \beta > 0$ *such that*

1. $|a(u,v)| \leq \alpha \|u\| \|v\| \quad u, v \in V.$
2. $\beta \|u\|^2 \leq a(u,v) \quad u \in V.$

Finally, let $f : V \to \mathbb{R}$ be a bounded linear functional on V. Then there exists a unique element $u \in V$ such that $a(u,v) = f(v)$ for all $v \in V$.

Theorem 2 (Riesz Representation)**.** *Let $f : V \to \mathbb{R}$ be a linear and continuous functional. Then there is a unique element $u_f \in V$ such that $f(v) = \langle u_f, v \rangle$ for every $v \in V$.*

a) Prove that for every $u \in V$ there is a $w_u \in V$ such that $a(u,v) = \langle w_u, v \rangle$ for every $v \in V$.

b) Call $A : V \to V$ the operator sending u into such w_u so that $a(u,v) = \langle Au, v \rangle$ for every $v \in V$. Prove that A is linear and continuous.

c) Prove that A is also injective, i.e. $\|Au\| = 0$ if and only if $u = 0$.

d) Prove that given a sequence $\{u^k\}_{k \in \mathbb{N}} \subset V$ and called $Au^k = v^k$, if $\|v^k - v\| \to 0$ as $k \to \infty$ for a $v \in V$, then there is $u \in V$ such that $Au = v$, i.e. range(A) is closed in V since $v \in$ range(A).

e) Prove that range$(A) = V$ (Hint: suppose it is not, then take a $v \neq 0$ that is in range$(A)^\perp = \{\bar{v} : \ \langle \bar{v}, Au \rangle = 0, \ \forall u \in V\}$ and reach a contradiction).

f) Show there is a $w \in V$ such that $f(v) = \langle w, v \rangle$ for every $v \in V$ and then conclude the proof of the theorem.

3 Consider the Poisson equation

$$\begin{cases} -u''(x) = 2, & x \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

a) Find the analytical solution u of this boundary value problem.

b) Write the weak formulation of the problem, specifying functional space V, linear form $f : V \to \mathbb{R}$ and bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$.

c) Consider now the finite dimensional space

$$V_h = \left\{ \varphi_k(x) := \frac{\sqrt{2}\sin((2k+1)\pi x)}{\pi(2k+1)} : \ k = 1, ..., n \right\}.$$

Prove that the basis functions are orthonormal with respect to the inner product defined by the bilinear form, i.e.

$$a(\varphi_k, \varphi_p) = \delta_{kp} = \begin{cases} 1, & k = p \\ 0, & k \neq p. \end{cases}$$

d) Define the Ritz-Galerkin formulation of the problem in the space $V_h \subset V$.

e) Find the solution $u_h \in V_h$ to this finite dimensional problem.

f) Verify Galerkin Orthogonality, i.e. that for every $v_h \in V_h$ it holds true

$$a(u - u_h, v_h) = 0.$$