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TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Fall 2021

Solutions to exercise set 2

1 Recall that Gaussian quadrature based on the nodes $\{\xi_1, ..., \xi_m\}$, $x_i \in [-1, 1]$, and weights $\{\bar{\omega}_1, ..., \bar{\omega}_m\}$ defines the following approximation

$$\int_{-1}^{1} f(\xi)d\xi \approx \sum_{i=1}^{m} \bar{\omega}_i f(\xi_i).$$

If we want to define a similar setting on the reference interval $[0, 1]$, we can map $[0, 1]$ into $[-1, 1]$ thanks to $\xi(x) = -1 + 2x$, with $x \in [0, 1]$. Thus, $x = \frac{\xi+1}{2}$ and

$$\int_0^1 f(x)dx = \frac{1}{2}\int_{-1}^{1} f\left(\frac{\xi+1}{2}\right)d\xi \approx \frac{1}{2}\sum_{i=1}^{m} \bar{\omega}_i f\left(\frac{\xi_i+1}{2}\right) = \sum_{i=1}^{m} \omega_i f(x_i)$$

if we set $\omega_i := \bar{\omega}_i/2$ and $x_i = \frac{\xi_i+1}{2}$. In this exercise, we will hence work with the weights ω_i and nodes x_i which are defined in this way.

a) If we consider the integral

$$\int_a^b f(u)du,$$

and define the affine change of variable $[0, 1] \ni x \mapsto a + hx = u \in [a, b]$ we get

$$\int_a^b f(u)du = h\int_0^1 f(a + hx)dx \approx h\sum_{i=1}^{m} \omega_i f(a + hx_i)$$

for nodes and weights defined as before.

b) Consider now

$$\int_a^b f'(u)g'(u)du$$

and set again $[0, 1] \ni x \mapsto a + hx = u = u(x) \in [a, b]$. By the chain rule, we have that

$$f'(u) = \frac{df}{du} = \frac{df}{dx}\frac{dx}{du} = \frac{1}{h}f'(a + xh)$$

which allows to conclude

$$\int_a^b f'(u)g'(u)du = h\int_0^1 \left(\frac{1}{h}f'(a + xh)\right)\left(\frac{1}{h}g'(a + xh)\right)dx \approx$$
$$\approx \frac{1}{h}\sum_{i=1}^{n} \omega_i f'(a + hx_i)g'(a + hx_i)$$

c) Recall that the mass and stiffness matrices, $\bar{M}$ and $\bar{K}$, on the master element $[0,1]$ are of the following form

$$\bar{M}_{rs} = \int_0^1 \hat{\varphi}_r(x)\hat{\varphi}_s(x)dx, \quad r,s \in 0,1,$$

$$\bar{K}_{rs} = \int_0^1 \hat{\varphi}'_r(x)\hat{\varphi}'_s(x)dx, \quad r,s \in 0,1.$$

By direct calculation, we can hence get

$$\bar{M} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad , \quad \bar{K} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

d) As for the previous point, just proceed by direct calculation of the integrals where, this time, $r,s \in \{0,1,2\}$. We then get

$$\bar{M} = \frac{1}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \quad , \quad \bar{K} = \frac{1}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

e) Even if the entries of $\bar{M}$ and $\bar{K}$ are easy to compute exactly, we can practice with Gaussian quadrature and verify the previously computed integrals are correct. Indeed, recall that if we use $m = 5$ quadrature nodes, we will get a quadrature formula which is exact for quartic polynomials, as those involved in these computations. We hence just need to apply the expression developed in the first point of this exercise, and get

$$M_{rs} = \int_0^1 \hat{\varphi}_r(x)\hat{\varphi}_s(x)dx = \sum_{i=1}^{5} \omega_i \hat{\varphi}_r(x_i)\hat{\varphi}_s(x_i),$$

$$K_{rs} = \int_0^1 \hat{\varphi}_r(x)\hat{\varphi}_s(x)dx = \sum_{i=1}^{5} \omega_i \hat{\varphi}_r(x_i)\hat{\varphi}_s(x_i).$$

You can find a simple implementation in this notebook `https://github.com/dadeslam/FEMCodesTMA4220/blob/main/Solution%20of%20Exercise%201%20-%20Exercise%20set%202.ipynb`

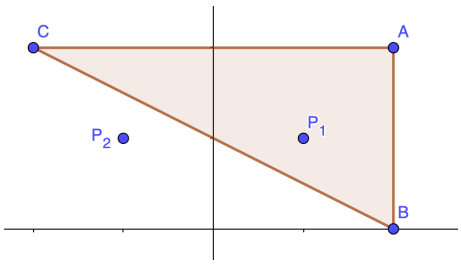


Figure 1: Triangle and points of interest

2 **a)** We recall that the barycentric coordinates $(u, v, 1-u-v)$ are the coefficients of the weighted sum of the vertices of the triangle that return the point of interest. More precisely, we should find $u, v \in \mathbb{R}$ such that

$$P = uA + vB + (1-u-v)C$$

for the two points P_1 and P_2. We start with the first:

$$\begin{cases} 1/2 = u + v - (1 - u - v) = 2u + 2v - 1 \\ 1/2 = u + (1 - u - v) = 1 - v \end{cases}$$

that gives $v = 1/2$ and $u = 1/4$. Thus we have the triple $(1/4, 1/2, 1/4)$. For the second point we have

$$\begin{cases} -1/2 = 2u + 2v - 1 \\ 1/2 = 1 - v \end{cases}$$

which gives $v = 1/2$, $u = -1/4$. Thus the triple associated to P_2 is $(-1/4, 1/2, 3/4)$.

b) Barycentric coordinates associated to a triangle allow to express an arbitrary point $P \in \mathbb{R}^2$ as a weighted sum of the vertices. We recall that a triangle is the convex-hull of its vertices, thus every point inside the triangle can be expressed as

$$P = \alpha_1 A + \alpha_2 B + \alpha_3 C,\ \alpha_i > 0,\ \alpha_1 + \alpha_2 + \alpha_3 = 1.$$

Being the barycentric coordinates a unique triple, we have that a point $P \in \mathbb{R}^2$ lies in the triangle ABC if and only if its barycentric coordinates are positive. We also notice that a point is on the boundary of the triangle if all the coordinates are non-negative and one of them is 0.

c) Let us denote with T the triangle defined by A, B and C. We introduce the map $(x, y) = uA + vB + (1 - u - v)C = \Phi(u, v)$. We have seen that $(x, y) \in T$ if and only if $0 \leq u, v \leq 1$ and also $1 - u - v \geq 0$. Thus it follows that also $u + v \leq 1$. This allows us to apply the following change of variable to the integral

$$\int_T f(x, y) dx dy = \int_0^1 \int_0^{1-u} f(\Phi(u, v)) |\det(J\Phi)| dv du$$

We now compute the Jacobian of Φ:

$$\Phi(u, v) = uA + vB + (1 - u - v)C = C + u(A - C) + v(B - C) \implies J\Phi(u, v) = [A - C, B - C].$$

We recall that the modulus of the determinant of this Jacobian matrix corresponds to 2 times the area of the triangle T. Thus the original integral writes:

$$\int_T f(x, y) dx dy = 2\text{Area}(T) \int_0^1 \int_0^{1-u} f(\Phi(u, v)) dv du.$$

d) We now focus on the particular case in which $f(x, y) = x + y$. What needs to be done is to compute the area of our triangle T and to evaluate $f(\Phi(u, v))$. We start with the latter task:

$$f(\Phi(u, v)) = 2u + 2v - u = u + 2v.$$

The area of the triangle is $\text{Area}(T) = 1$. This means that the integral is

$$\begin{aligned} \int_T (x + y) dx dy &= 2 \int_0^1 \int_0^{1-u} (u + 2v) dv du = 2 \int_0^1 \left(u(1 - u) + (1 - u)^2 \right) du = \\ &= 2 \left(\frac{u^2}{2} - \frac{u^3}{3} - \frac{(1 - u)^3}{3} \right) \Big|_{u=0}^1 = 2 \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} \right) = 1. \end{aligned}$$

3 In order to define N uniformly wide elements in the interval $[a, b]$, we define a triangulation $\mathcal{T}_h$

$$a = c_0 < c_1 < ... < c_N = b$$

made by $N + 1$ equally spaced nodes, where $h = c_i - c_{i-1} = (b - a)/N$. Thus, if $e_i = [c_{i-1}, c_i]$ is the $i-$th element of $\mathcal{T}_h$, then the local mass and stiffness matrices, M_{e_i} and K_{e_i}, can be obtained from those on the reference element $[0, 1]$, denoted with $\bar{M}$, $\bar{K}$, as

$$M_{e_i} = h\overline{M} \quad K_{e_i} = \frac{1}{h}\overline{K}.$$

The elements are C^0-continuous, so the overlapping is at the first and last entries of each local matrix, and the dimension of the global system matrix is $(Np+1)\times(Np+1)$ where p is the degree of the polynomial basis functions.

For the local element load vectors, we need to apply Gaussian quadrature. Indeed, we set

$$f_{e_i}^{(r)} = h\int_0^1 \hat{\varphi}_r(\xi) f(c_i + h\xi)d\xi \approx h\sum_{j=1}^{5} w_j \hat{\varphi}_r(x_j) f(c_i + x_j h) =$$

$$= h\sum_{j=1}^{5} w_j \varphi_{e_{ir}}(c_i + x_j h) f(c_i + x_j h), \quad r \in \{0, .., p\}$$

and define the element load vector $f_{e_i} := \left[f_{e_i}^{(0)}, ..., f_{e_i}^{(p)}\right]^T$.

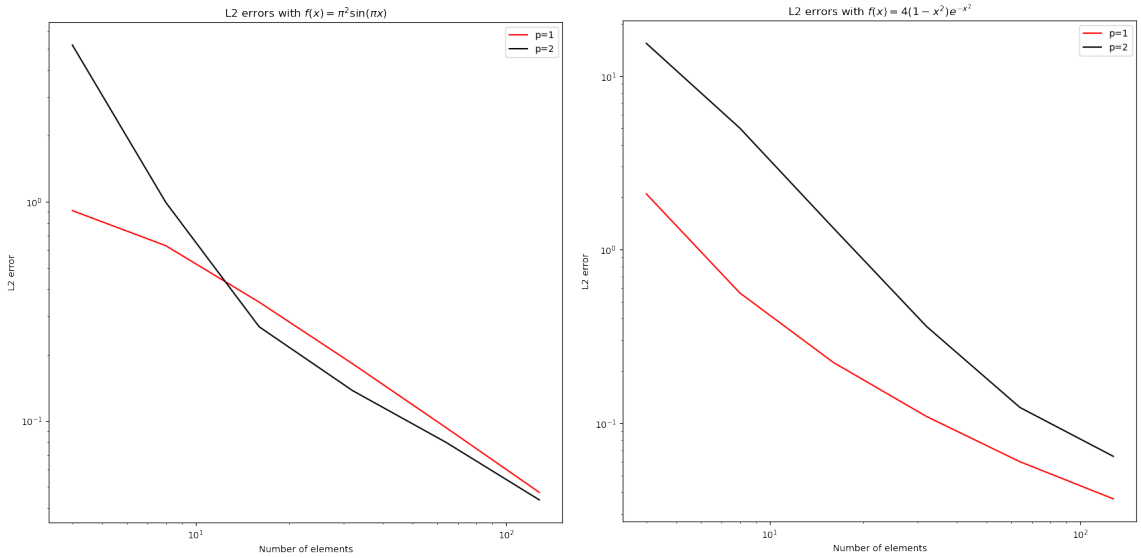
For the calculation of the global error, we need again Gaussian quadrature. To do so, we work on each element and consider the basis functions which are not zero on this element. In other words, we consider the restriction of the solution $u_h(x) = \sum_{k=0}^{N} u_k \varphi_k(x)$ to each element e_i. The error on this element thus amounts to

$$E_i^2 = \int_{e_i} (u(x) - u_h(x))^2 dx = h\int_0^1 \left(u(c_i + hx) - \sum_{k=0}^{p} u_k \varphi_k(c_i + hx)\right)^2 dx \approx h\sum_{j=1}^{m} \omega_j h_i(x_j)$$

where

$$h_i(x_j) = \left(u(c_i + hx_j) - \sum_{k=0}^{p} u_k \varphi_k(c_i + hx_j)\right)^2.$$

It then follows $E = \sqrt{\sum_{i=0}^{N-1} E_i^2}$. You can find a proposal for the implementation here `https://github.com/dadeslam/FEMCodesTMA4220/blob/main/Solution%20of%20Exercise%202%20-%20Exercise%20set%202.ipynb` and you can see below the convergence plots for the two test cases mentioned in the exercise.



(a) Case with $u(x) = \sin(\pi x)$ (b) Case with $u(x) = e^{-x^2}$

Figure 2: Convergence plot of the simulations