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TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods
Fall 2022

Solutions to exercise set 3

1 **Remark** The solution proposed here motivates in a constructive way the final expressions obtained for the functions. However, a more systematic solution could rely on the following lemma, that is actually slightly hidden in the reasoning we illustrate in the solution.
Lemma (Brenner Scott page 71): Let p be a polynomial of degree d defined on $\mathbb{R}^n$. Let $\mathcal{A} = \{x \in \mathbb{R}^n : \; L(x) = 0\}$ be a hyperplane where p vanishes. Then there is a polynomial q of degree $d-1$ such that $p = Lq$.

Just to make an example of how this lemma leads to solving this problem, we consider the function $\hat{\varphi}_3$. This is a polynomial of degree $d = 3$. Such a polynomial needs to vanish for $\lambda_3 = 0$. Indeed this value corresponds to the segment connecting $P1$ with $P2$, and a cubic polynomial vanishing on 4 points is identically 0. So we can write $\hat{\varphi}_3 = \lambda_3 p$ for a quadratic polynomial p. Similarly $\hat{\varphi}_3$ needs to vanish for $\lambda_3 = 1/3$ and $\lambda_3 = 2/3$. This leads to $\hat{\varphi}_3 = a\lambda_3(\lambda_3 - 1/3)(\lambda_3 - 2/3)$. One can now conclude imposing $\hat{\varphi}_3(0,0,1) = 1$.

We now reason in a more constructive way, without directly relying on the lemma above, even if the main motivation is still that result. We first start with the basis functions that are 1 at the corners, which we denote with $\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3$. We focus on $\hat{\varphi}_1$ and then extend to the others by symmetry. We notice that such a function needs to be 0 for all the values of λ_2 and λ_3. This means that none of these two variables can appear in the expression. Furthermore, we point out that also for $\lambda_1 = 0$, $\lambda_1 = 1/3$ and $\lambda_1 = 2/3$ we should get a 0 value. This leads to a function of the form

$$\hat{\varphi}_1(\lambda_1, \lambda_2, \lambda_3) = a_1\lambda_1(\lambda_1 - 1/3)(\lambda_1 - 2/3).$$

We conclude imposing

$$\hat{\varphi}_1(1,0,0) = \frac{2a_1}{9} = 1 \implies a_1 = \frac{9}{2}$$

and hence we get

$$\hat{\varphi}_i = \frac{9}{2}\lambda_i\left(\lambda_i - \frac{1}{3}\right)\left(\lambda_i - \frac{2}{3}\right), \quad i = 1,2,3.$$

We now work on $\hat{\varphi}_4$ and then extend by symmetry to the remaining functions associated to nodes on the boundary of the element. $\hat{\varphi}_4(\lambda_1, \lambda_2, \lambda_3)$ has to satisfy $\hat{\varphi}_4(0, 1/3, 2/3) = 1$. Moreover, it has to be 0 whatever the choice of λ_1 we make.

Thus, λ_1 can not appear in the expression. Moreover, $\hat{\varphi}_4$ needs to be 0 on the nodes along the segment connecting P_1 with P_2 and P_6 with P_5, i.e. on $(\lambda_1, 1-\lambda_1, 0)$ and $(\lambda_1, 2/3-\lambda_1, 1/3)$ for suitable choices of $\lambda_1 \in \{0, 1/3, 2/3, 1\}$. Thus λ_3 and $\lambda_3 - 1/3$ need to be factors of the $\hat{\varphi}_4$ polynomial. Moreover, the only vertex that remains out of the analysis is the one with $\lambda_2 = 0$, and hence λ_2 needs to be a factor too. This allows to say that $\hat{\varphi}_4 = d_4\lambda_2\lambda_3(\lambda_3 - 1/3)$. To compute d_4, we impose $\hat{\phi}_4(0, 1/3, 2/3) = d_4 2/9(1/3) = 1$ and hence $d_4 = 27/2$. This allows to conclude

$$\hat{\phi}_4 = \frac{27}{2}\lambda_2\lambda_3(\lambda_3 - 1/3).$$

By symmetry we can obtain the expression for the other basis functions:

$$\begin{aligned}
\hat{\varphi}_5 &= 27/2\lambda_2\lambda_3\left(\lambda_2 - 1/3\right)\\
\hat{\varphi}_6 &= 27/2\lambda_3\lambda_1\left(\lambda_1 - 1/3\right)\\
\hat{\varphi}_7 &= 27/2\lambda_3\lambda_1\left(\lambda_3 - 1/3\right)\\
\hat{\varphi}_8 &= 27/2\lambda_1\lambda_2\left(\lambda_2 - 1/3\right)\\
\hat{\varphi}_9 &= 27/2\lambda_1\lambda_2\left(\lambda_1 - 1/3\right).
\end{aligned} \tag{1}$$

Moerover, since the remaining function $\hat{\varphi}_{10}$ needs to vanish on all the vertices on the boundary of the element, we conclude that the only possiblity is $\hat{\varphi}_{10} = 27\lambda_1\lambda_2\lambda_3$.

2

a) We first recall that for a function of the only x_1, it holds $u(x_1) = \int_{-M}^{x_1} u'(s)ds$ if $u(-M) = 0$. We thus can say, fixing x_2, that

$$\begin{aligned}
|u(x_1, x_2)|^2 &= \left|\int_{-M}^{x_1} \partial_{x_1} u(s, x_2) ds\right|^2 \leq \\
&\leq \int_{-M}^{x_1} ds \cdot \int_{-M}^{x_1} |\partial_{x_1} u(s, x_2)|^2\, ds
\end{aligned}$$

where the last inequality is the Cauchy-Schwartz one. We can now conclude that

$$|u(x)|^2 \leq C\int_{-M}^{M}\left|\frac{\partial u}{\partial x_1}(y_1, x_2)\right|^2 dy_1, \qquad \text{for all } x = (x_1, x_2) \in \Omega$$

where $C = 2M$.

b) We now integrate against x_2 the estimate obtained in the previous point:

$$\int_{-M}^{M} |u(x_1, x_2)|^2 dx_2 \leq C\int_{-M}^{M}\int_{-M}^{M} |\partial_{x_1} u(x_1, x_2)|^2 dx_1 dx_2 = C\int_\Omega \left|\frac{\partial u}{\partial x_1}(x)\right|^2 dx$$

for all $x_1 \in [-M, M]$. The last equality comes from Fubini-Tonelli's theorem.

c) We now integrate again, this time over x_1, the previously obtained estimate to get

$$\begin{aligned}
\|u\|^2_{L^2(\Omega)} &= \int_\Omega |u(x)|^2 dx \leq 2M\int_{-M}^{M}\int_\Omega\left|\frac{\partial u}{\partial x_1}(x)\right|^2 dx dx_1\\
&= 4M^2\int_\Omega\left|\frac{\partial u}{\partial x_1}(x)\right|^2 dx \leq 4M^2\int_\Omega\left(\left|\frac{\partial u}{\partial x_1}(x)\right|^2 + \left|\frac{\partial u}{\partial x_2}(x)\right|^2\right) dx = \\
&= 4M^2|u|^2_{H^1(\Omega)}.
\end{aligned}$$

d) Being Ω bounded, we can always find a superset of the form $[-M, M]^2$ since the diameter of Ω is finite. Let $u \in H_0^1(\Omega)$. We can then rely on an extension of u to $[-M, M]^2$ defined as

$$\bar{u}(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

This function satisfies the requirements. This extension allows to prove the Poincaré inequality for a generic $\Omega \subset \mathbb{R}^2$ since

$$\|u\|^2_{L^2(\Omega)} = \|\bar{u}\|^2_{L^2([-M,M]^2)} \leq_{\text{Point (c)}} 4M^2|\bar{u}|^2_{H^1([-M,M]^2)} = 4M^2|u|^2_{H^1(\Omega)}$$

as desired.

We notice that the reasoning done in the exercise extends, without any complication, to the generic case of $\Omega \subset \mathbb{R}^d$, simply iterating the procedure up to exhausting the dimension d. Moreover, it is important to remark that the constant C in the estimate, grows as the diameter of the domain of interest.

For the sake of completeness, we notice that Poincaré inequality can be applied iteratively to obtain estimates involving higher order weak derivatives. More precisely, if we have $u \in H_0^1(\Omega)$ for any $k \geq 1$, then $\partial_{x_1} u \in H_0^{k-1}(\Omega)$ for any $i = 1, ..., d$. Thus we can get

$$\|u\|_{L^2(\Omega)} \leq C|u|_{H^1(\Omega)} \leq \cdots \leq C|u|_{H^k(\Omega)}, \forall u \in H_0^k(\Omega).$$

e) Let $z_1, ..., z_5$ be the nodes. We start considering just the 4 boundary nodes z_1, z_2, z_3, z_4, and we work with the reference element $[0, 1]^2$. The basis functions

$$\Phi_1(x, y) = (1 - x)(1 - y), \quad \Phi_2(x, y) = x(1 - y),$$

$$\Phi_3(x, y) = xy, \quad \Phi_4(x, y) = (1 - x)y$$

fully determine the space $\mathcal{Q}_1 = \{\sum_j c_j p_j(x) q_j(y) : p_j, q_j \in \mathcal{P}_1\}$ where $\mathcal{P}_1$ is the space of one-variable polynomials of degree at most 1. Then, we define the bubble function $\Phi_5(x, y) = 16xy(1 - x)(1 - y)$ that is valued 1 on the node z_5 and 0 on all the edges. We then introduce 4 scalar coefficients $\alpha_1, ..., \alpha_4 \in \mathbb{R}$ that have the aim of constructing a compatible finite element involving all the 5 nodes. Let $\hat{\Phi}_i(x, y) = \Phi_i(x, y) - \alpha_i \Phi_5(x, y)$ for $i = 1, ..., 4$. Moreover, the basis functions have to satisfy the partition of unity and hence we have to have

$$\sum_{i=1}^{4} \hat{\Phi}_i + \Phi_5 = \sum_{i=1}^{4} \Phi_i + (1 - \sum_{i=1}^{4} \alpha_i)\Phi_5 = 1 + (1 - \sum_{i=1}^{4} \alpha_i)\Phi_5 = 1.$$

This implies that $\sum_{i=1}^4 \alpha_i = 1$. Moreover, we already have that $\hat{\Phi}_i(z_i) = 1$, but it is not true in general that $\hat{\Phi}_i(z_5) = 0$ for $i = 1, ..., 4$. We hence impose this condition to recover the finite element space. By symmetry, we get that

$$\hat{\Phi}_i(1/2, 1/2) = 1/4 - \alpha_i = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{4}.$$

We notice that these coefficients satisfy the partition of unity property and this concludes the derivation. Thus, the nodes $z_1, ..., z_5$ and the functions $\hat{\Phi}_1, ..., \hat{\Phi}_4, \Phi_5$ define a compatible finite element.

A different solution can be to reconduce this setting to the known one in which the square is triangulated in 4 triangles, with shared internal vertex. In this case, linear polynomials are fully determined by the 3 degrees of freedom (i.e. the 3 vertices). Thus, we have that linear polynomials are fully determined by $\mathcal{N} = \{N_1, ..., N_5\}$, where $N_i(z_j) = \delta_{ij}$ and they are linear.

3 Consider the two-dimensional steady heat equation

$$\begin{aligned} -\nabla(\kappa\nabla u) &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega_D \\ H(u) &= \bar{t} \text{ on } \partial\Omega_N. \end{aligned}$$

a) (See, e.g., A. Quarteroni pag. 48) We now establish the weak formulation of the problem integrating the equation against a test function $v \in X$, for some test space X we determine later:

$$\int_\Omega -\nabla \cdot (k\nabla u) v dx = \int_\Omega f v dx.$$

By Green's theorem and the imposition of the boundary conditions, this turns out to be

$$a(u, v) = \int_\Omega k\nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_{\partial\Omega_N} v(k\nabla u) \cdot n d\Gamma.$$

We now introduce the Neumann operator

$$H(u) = k\nabla u \cdot n = k\frac{\partial u}{\partial n} = \bar{t} \text{ on } \partial\Omega_N$$

and conclude

$$l(v) = \int_\Omega f v dx + \int_{\partial\Omega_N} v\bar{t} d\Gamma.$$

Since the weak formulation involves up to first order weak derivatives, we require $u, v \in H^1(\Omega)$. Moreover, having Dirichlet boundary conditions on $\partial\Omega_D$, we set $X = \{w \in H^1(\Omega) : w = 0 \text{ in the sense of traces on } \partial\Omega_D\}$. This is our test and solution space, since we have homogeneous Dirichlet boundary conditions.

b) (See, e.g., A. Quarteroni pag. 116,117 ex. 7-8) First of all we notice that if $k(x, y) \equiv \bar{k} \in \mathbb{R}$ we need to assume $\partial\Omega_D \neq \emptyset$, otherwise even when the solution exists, it will be just defined up to a constant and hence not unique. Furthermore, we assume $f \in L^2(\Omega)$, $k \in L^\infty(\Omega)$ (for example it can be continuous on the compact set Ω), $\bar{t} \in L^2(\partial\Omega_N)$. We derive the existence and uniqueness of the solution via Lax-Milgram theorem. Indeed, a is coercive since

$$a(u, u) = \int_\Omega k\|\nabla u\|^2 dx \geq k_{min}|u|^2_{H^1(\Omega)} \geq C\|u\|^2_{H^1}$$

where the last step comes from Poincaré inequality. It is continuous since

$$a(u, v) \leq \|k\|_{L^\infty(\Omega}|v|_{H^1}|u|_{H^1} \leq \|k\|_{L^\infty(\Omega)}\|v\|_{H^1}\|u\|_{H^1}.$$

The functional l is linear. Moreover, it is bounded because of the assumptions made on $\bar{t}$ and f. We hence conclude existence and uniqueness by Lax-Milgram theorem.

c) We need to fix a finite dimensional space

$$X_h = \text{span}\{\varphi_i : \Omega \to \mathbb{R} : \ i = 1, ..., n\} \subset X$$

where the solution will be approximated. Thus, we look for a function

$$u_h = \sum_{i=1}^{n} u_h^i \varphi_i(x) \in X_h$$

such that, for any $v_h \in X_h$, it satisfies

$$a(u_h, v_h) = l(v_h).$$

d) To handle inhomogeneous Dirichlet boundary conditions, still working with symmetric solution and test spaces, we lift the solution via a function $R_{\bar{u}} \in H^1(\Omega)$ which satisfies $R_{\bar{u}}|_{\partial\Omega_D} = \bar{u}$. Then, we define the solution $u = \mathring{u} + R_{\bar{u}}$ where $\mathring{u} \in X$, i.e. it is valued 0 on the Dirichlet boundary. In this way, looking for the solution $\mathring{u}$ we get the variational formulation: Find $\mathring{u} \in X$ such that, for any $v \in X$:

$$a(\mathring{u}, v) = l(v) - a(R_{\bar{u}}, v).$$

In the FEM code, this can be translated into working with $R_{\bar{u}} = \sum_{i \in \mathcal{B}_\mathcal{D}} \bar{u}^i \varphi_i(x)$, where $\mathcal{B}_D$ are the indices of the boundary nodes of the domain. Hence the i-th row of the FEM linear system has an additional contribution on the right hand side which is due to $-\sum_{j \in \mathcal{B}_D} a(\varphi_j, \varphi_i)\bar{u}^j$. The solution, will then be $u = \mathring{u} + R_{\bar{u}}$.

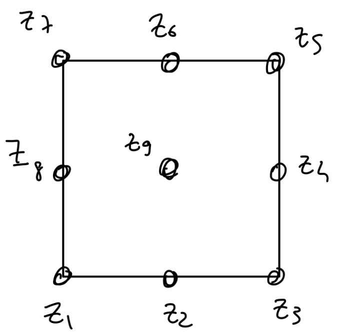

Figure 1: Labelling of the triangulation nodes

e) Suppose to number the nodes as in Figure 1, and hence let z_9 be the inner node in both the triangulations. Since the two triangulations have all nodes but one on the Dirichlet boundary, we can already say that the only unknown of the linear systems $Au = b$ will be u_9. Thus, we can decide to impose the boundary conditions removing the rows and columns of the stiffness matrix associated to boundary nodes and what survives is a scalar equation of the form $a_{99}u_9 = b_9$. We thus find b_9 and a_{99} for both the triangulations and then plot the diagonal profile of the solution. For the right hand side, we have

$$b_9 = f \int_\Omega \phi_9(x, y) dx dy$$

where $f \in \mathbb{R}$ is the constant scalar forcing term. This integral corresponds to the volume defined by the piecewise linear function ϕ_9. In the left triangulation, ϕ_9 defines a pyramid of basis given by the full square and hence $b_9 = f/3$, while for the right triangulation the basis has smaller area (given by the boundary nodes belonging to an element where z_9 is one of the vertices) and we get $b_9 = f/4$. For the left hand side, we have

$$a_{99} = k \int_{\Omega} \|\nabla \phi_9\|^2 dx = k \sum_{i=1}^{N} \int_{T_k} \|\nabla \phi_9\|^2 dx.$$

Recall that on a triangle of vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ the basis functions ϕ_1, ϕ_2, ϕ_3 that satisfy $\phi_i(z_j) = \delta_{ij}$ are defined as $\phi_i(x, y) = a_i x + b_i y + c_i$ where

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, for example on the triangle T_1 of vertices $(1/2, 1/2), (1, 1/2), (1, 1)$, ϕ_9 takes the form $\phi_9 = -2x + 2$. By symmetry, for the left triangulation, all the integrals take the same values and hence

$$a_{99} = 8k \int_{T_1} 4dx = 32k|T_1| = 4k.$$

This implies that for the left triangulation $u_9 = \frac{f}{12k}$. For the right triangulation, not all the integrals for the a_{99} terms coincide, in particular the triangles sharing z_9 and living on the bottom right and top left squares have different integrals. Let us call T_2 the one with vertices $(1/2, 1/2), (1, 1/2), (1/2, 0)$. Then we have

$$a_{99} = k \left(4 \int_{T_1} \|\nabla \phi_9\|^2 dx + 2 \int_{T_2} \|\nabla \phi_9\|^2 dx \right).$$

By previous computation we have $\phi_9|_{T_2} = -2x + 2y + 1$ and hence $\|\nabla \phi_9\|^2 = 8$. Thus

$$a_{99} = 4k$$

again. We conclude that for the right triangulation $u_9 = \frac{f}{16k}$. To conclude, since for the right triangulation $u = 0$ both on z_6 and z_8, and the same happens for z_2 and z_4 the solution being piecewise linear, will be 0 even on the edges connecting them. This fact and the different support of ϕ_9 brings to the two following different profiles of the solution:

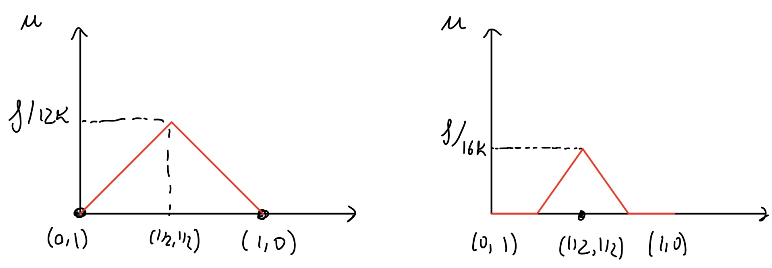

Figure 2: Diagonal profiles of the two solutions