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TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Fall 2021

Solutions to exercise set 1

1 **a)** **False**: Let $v \in V$. Then, for any $\mathbb{R} \ni \lambda \neq 1$, $\lambda v \notin V$.

b) **True**: Since $f, v \in L^2(0,1)$, we have

$$\|f\|_{L^2}^2 = \int_0^1 |f(x)|^2 dx < \infty, \quad \|v\|_{L^2}^2 = \int_0^1 |v(x)|^2 dx < \infty.$$

By Hölder's inequality we have that

$$\int_0^1 |f(x)v(x)|dx \leq \|f\|_p \|g\|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

and hence if $p = 2$ we get $\|fv\|_{L^1} \leq \|f\|_{L^2}\|v\|_{L^2} < \infty$.

c) **True**: Indeed being a continuous on a compact set, it attains a global minimum $m > 0$ on Ω. This means

$$b(u,u) = \int_\Omega a(x)u(x)^2 dx \geq m \int_\Omega u^2(x)dx = m\|u\|_{L^2}^2,$$

where the first inequality comes from the monotonicity of the integral.

d) **False**: Just take $u = c$ for any constant $c \in \mathbb{R}$ and you get $u'(x) \equiv 0$ and hence $|u|_{H^1(\Omega)} = 0$. On the other hand, if we impose $u \in H_0^1(\Omega)$, we get that this result becomes true. Indeed, by Poincaré inequality, when $u \in H_0^1(\Omega)$ the H^1 semi-norm becomes a norm.

2 **a)** We first do not specify V and take a test function $v \in V$ against which we integrate both the sides of the equation. Then we decide what is the correct space V based on the weak formulation we obtain.

$$-\int_I (a(x)u'(x))'v(x)dx = \int_I f(x)v(x)dx.$$

We now apply the integration by parts and the boundary conditions to derive the bilinear form b.

$$\begin{aligned}
&-\int_I (a(x)u'(x))'v(x)dx = -\big(a(x)u'(x)v(x)\big)\,|_{x=0}^{x=L} + \int_I a(x)u'(x)v'(x)dx \\
&= -\big(a(L)u'(L)v(L)\big) + \big(a(0)u'(0)v(0)\big) + \int_I a(x)u'(x)v'(x)dx \\
&= k_L(u(L) - g_L)v(L) + k_0(u(0) - g_0)v(0) + \int_I a(x)u'(x)v'(x)dx.
\end{aligned}$$

We now keep on the left hand side just the terms involving the unknown u and get

$$F(v) = \int_I fv dx + k_0 g_0 v(0) + k_L g_L v(L),$$
$$b(u,v) = k_L u(L)v(L) + k_0 u(0)v(0) + \int_I au'v'dx.$$

We notice that the least amount of regularity we need to have for the integrals to be defined is $u', v' \in L^2(I)$. For this reason the functional space where we define the weak formulation is $V = H^1(I)$. Notice that we do not make any assumption on the boundary values of u, v and hence we can not work on H^1_0.

b) The bilinear form is symmetric since it is easy to verify that $b(u,v) = b(v,u)$ for every $u, v \in V$.

c) The bilinear form is continuous. We assumed a continuous, thus it is in particular L^∞ on I. This brings us to the first bound

$$\left|\int_I au'v'dx\right| \leq \|au'v'\|_1 \leq \|a\|_{L^\infty}\|u'v'\|_{L^1}$$
$$\leq \|a\|_{L^\infty}\|u'\|_{L^2}\|v'\|_{L^2} \leq \|a\|_{L^\infty}\|u\|_V\|v\|_V.$$

To bound the remaining part, we recall that in 1 dimension the embedding of V into C^0 is continuous. Thus

$$|ku(s)v(s)| \leq \sup_{x\in I}|ku(x)v(x)| = \|kuv\|_{C^0(I)} \leq k\|u\|_{C^0(I)}\|v\|_{C^0(I)}$$
$$\leq c\|u\|_V\|v\|_V.$$

Hence b is continuous.

d) The linear form is continuous. Indeed

$$\left|\int_I fv dx\right| \leq \|f\|_{L^2}\|v\|_V$$

and the other part can be bounded again using the continuity of the trace operator, leading to $|F(v)| \leq d\|v\|_V$ as desired.

3 **a)** 1. We first compute the $L^2([-1,1])$ norm

$$\int_{-1}^1 (|x|^\alpha)^2\, dx = 2\int_0^1 |x|^{2\alpha}dx = \frac{2}{2\alpha+1}\left[x^{2\alpha+1}\right]_0^1.$$

We notice that this blows up at zero if $2\alpha + 1 < 0$. If $2\alpha + 1 = 0$, i.e. $\alpha = -1/2$, the function we integrate becomes $1/|x|$, and hence the integral is $\log x$. This blows up at $x = 0$ too. Thus, we conclude that $|x|^\alpha \in L^2([-1,1])$ if $\alpha > -\frac{1}{2}$.

2. We can proceed similarly as before, but considering the improper integral

$$\int_1^\infty |x|^{2\alpha}dx = \lim_{s\to+\infty}\int_1^s |x|^{2\alpha}dx = \lim_{s\to\infty}\frac{1}{2\alpha+1}[x^{2\alpha+1}]_1^s.$$

This time, the problem is when the function is unbounded at infinity, i.e. when $2\alpha + 1 \geq 0$. The remaining case, instead, gives a bounded result and hence $|x|^\alpha \in L^2([1,+\infty))$ when $\alpha < -1/2$.

3. Working in a radially symmetric domain, we pass to polar coordinates for the point $x = (x_1, x_2)$: $x_1 = r\cos s$, $x_2 = r\sin s$. Thus, $f(x_1(r,s), x_2(r,s)) = x_1^{2\alpha} + x_2^{2\alpha} = r^{2\alpha}$. The integral, in polar coordinates, turns into

$$\int_{B_1(0)} |x|^{2\alpha} dz = \int_0^{2\pi} \int_0^1 r^{2\alpha+1} dr ds = 2\pi \int_0^1 r^{2\alpha+1} dr = \frac{2\pi}{2\alpha+2} r^{2\alpha+2}\Big|_{r=0}^1.$$

The resulting integral, is indeed similar to the one of point (1). We thus conclude, $2\alpha + 2 > 0$, i.e. $\alpha > -1$.

b) 1. By Weierstrass Theorem, we know that continuous functions are bounded on compact sets, i.e. on closed and bounded subsets of $\mathbb{R}$. This means $|f(x)| < M$ for any $x \in D$ and hence $\|f\|_{L^2(D)} \leq M^2 \mu(D)$, where $\mu(D)$ is the measure of D.

2. Continuous functions $f \in C^0(D)$ do not live necessarily in $H^1(D)$. As an example, just consider $f(x) = \sqrt{x} \in C^0([0,1])$ and for which $f'(x) = \frac{1}{2\sqrt{x}}$ is not in $L^2([0,1])$.

c) 1. Assume there are two weak derivatives v_1, v_2. We now prove they coincide in the L^2 sense, i.e. $\|v_1 - v_2\|_{L^2} = 0$.

$$\int_\Omega (v_1 - v_2)\phi dx = \int_\Omega (u - u)\phi' dx = 0.$$

Being L^2 an inner product space, and having $\langle v_1 - v_2, \phi\rangle_{L^2} = 0$ for any $\phi \in C_c^\infty(\Omega)$, allows to conclude (by density) that $\|v_1 - v_2\|_{L^2} = 0$. Thus, the weak derivative is unique.

2. When a strong derivative does exist, i.e. $u \in C^1$, we have that by integration by parts follows

$$\int_\Omega u\phi' dx = [u\phi]|_{\partial\Omega} \; - \int_\Omega u'\phi dx = -\int_\Omega u'\phi dx.$$

We thus can conclude, by uniqueness of the weak derivative, that u' is the weak derivative of u. We recall that $[u\phi]|_{\partial\Omega}$ vanishes since ϕ is compactly supported in Ω, and hence it is valued 0 at $\partial\Omega$.

d) 1. We start by computing the two L^2 norms.

$$\int_1^3 f_1(x)^2 dx = \int_1^2 (x-1)^2 dx + \int_2^3 dx = 1 + \frac{(x-1)^3}{3}\Big|_1^2 = \frac{4}{3} < \infty \implies f_1 \in L^2((0,3)),$$

$$\int_0^3 f_2(x)^2 dx = \int_0^1 x^2 dx + \int_1^3 4dx = \frac{1}{3} + 8 = \frac{25}{3} < \infty \implies f_2 \in L^2((0,3)).$$

2. We now check that $f_1 \in H^1(\Omega)$, first noticing that it is continuous so it can be H^1. Its weak derivative comes from

$$\int_0^3 f_1(x)\phi'(x)dx = \int_1^2 (x-1)\phi'(x)dx + \int_2^3 \phi'(x)dx =$$
$$= [(x-1)\phi(x)]|_1^2 + \phi(3) - \phi(2) - \int_1^2 \phi(x)dx.$$

We recall that $\phi(3) = 0$ since ϕ is compactly supported. Thus just remains

$$-\int_1^2 \phi(x)dx,$$

which allows to conclude that the weak derivative of f_1 is

$$v_1(x) = \begin{cases} 1, & x \in (1,2), \\ 0, & \text{otherwise.} \end{cases}$$

3. We easily conclude that f_2 is not in H^1 since H^1 functions defined on $\mathbb{R}$ are continuous, while f_2 is not.
4. Similarly, since being $H^2(\Omega)$ implies having $v_1 \in H^1$, we should at least have $v_1(x) \in C^0(\Omega)$. This is not the case, thus $f_1 \notin H^2(\Omega)$.

4 You can find a notebook with a proposal for the numerical implementation of these requests here `https://github.com/dadeslam/FEMCodesTMA4220/blob/main/Solution%20Ex%204%20-%20Exercise%20set%201.ipynb`

a) The stiffness matrix is built with a loop where we add, at each iteration, a new contribution to the matrix. This contribution is due to the shared supports of the basis functions $\varphi_i(x)$ and $\varphi_{i+1}(x)$, and it amounts to the L^2 product of their weak derivatives over the element $[x_i, x_{i+1}]$. Indeed, the matrix

$$B = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

is precisely given by these integrals

$$\int_{x_i}^{x_{i+1}} \varphi_k'(x)\varphi_j'(x)dx$$

with $k, j \in \{i, i+1\}$. To compute them, we can refer to the master element $\Omega = [0,1]$ and to the basis functions $\hat{\varphi}_0(\xi) = 1 - \xi$, $\hat{\varphi}_1(\xi) = \xi$. More precisely

$$B_{rs} = \int_{x_i}^{x_{i+1}} \varphi_k'(x)\varphi_j'(x)dx = \frac{1}{h}\int_0^1 \hat{\varphi}_r'(\xi)\hat{\varphi}_s'(\xi)d\xi = \frac{1}{h}(-1)^{r+s},$$

$r, s \in \{0, 1\}$, just by direct computations.

b) For the code just look at the notebook linked at the top of this exercise. The way to derive what to change in this case, comes from the weak formulation of the problem. Since we just have information on the value of the solution u at $x = 0$, we define the space in which the test function v lives as

$$\{w \in H^1(0,1) : w(0) = 0\}.$$

Thus, the weak formulation writes

$$\int_0^1 fvdx = \int_0^1 vdx = \int_0^1 -u_{xx}vdx = -[u_x v]|_0^1 + \int_0^1 u_x v_x dx =$$
$$= -u_x(1)v(1) + u_x(0)v(0) + \int_0^1 u_x v_x dx$$

i.e.

$$\int_0^1 u_x v_x dx = u_x(1)v(1) + \int_0^1 v dx = v(1) + \int_0^1 v dx.$$

If we pass to the Galerkin problem based on linear basis functions, we get that the $i-$th row of the linear system reads

$$\sum_{j=1}^{n} u_j \int_0^1 \varphi_j'(x)\varphi_i'(x)dx = \varphi_i(1) + \int_0^1 \varphi_i(x)dx.$$

Since $\varphi_i(1) \neq 0$ if and only if $i = n$, we conclude that what should be changed with respect to the Dirichlet problem is

- We first have to remove the changes done in the last row of the stiffness matrix A,
- Then, we should add to the right hand side $\varphi_n(1) = 1$ in the last entry of the load vector.

We notice hence what does it mean to have natural boundary conditions in this setting, since the imposition of this Neumann condition does not come from the choice of the solution/test space, but on the definition of the weak formulation (in particular of the linear functional $F(v)$). As a summary, we now get

$$Au = f + [0, ..., 0, 1]^T.$$

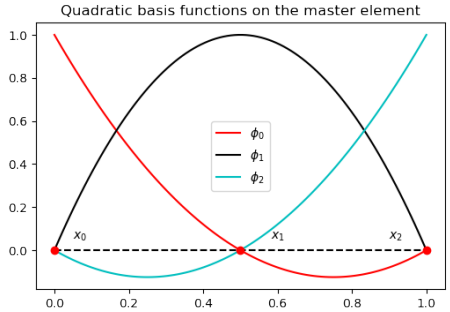
c) To solve exactly the ODE, we integrate both sides two times, imposing then the boundary conditions:

$$u'(x) = A - x,$$

$$u(x) = B + Ax - \frac{x^2}{2}$$

so $u'(1) = 1 = A - 1$ thus $A = 2$, and $u(0) = 0 = B$. It follows $u(x) = 2x - \frac{x^2}{2}$.

d) A proposal for the code to solve this point is given in the notebook linked on top. The reasoning here is similar to the one done for linear elements, however since quadratic polynomials have three degrees of freedom, to fully determine a basis of the space X_h^2, we need to add one internal node for each element. Thus, on the master element $[0, 1]$ we get the reference basis functions presented in the figure here. We denote with odd indices the intermediate nodes x_{2i+1}, while



with even ones x_{2i} the vertices of the elements. Here $h = x_{2i+2} - x_{2i}$. As for the increments in the stiffness matrix for linear elements, here we need to compute

them for the penta-diagonal stiffness matrix of this problem. The reasoning is the same, hence based on computing

$$B_{rs} = \int_{x_{2i}}^{x_{2i+2}} \varphi_k'(x)\varphi_j'(x)dx = \frac{1}{h}\int_0^1 \hat{\varphi}_r'(\xi)\hat{\varphi}_s'(\xi)d\xi,$$

with $r, s \in \{0, 1, 2\}$ and $k, j \in \{2i, 2i+1, 2i+2\}$. Just by computing these integrals exactly, we get

$$B = \frac{1}{3h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}.$$

In the code, we will hence loop over the elements and compute the contribution of each specific element to the stiffness matrix. The same needs to be done for the load vector, where we have increments of the form

$$b_r = \int_{x_{2i}}^{x_{2i+2}} \varphi_k(x)dx = h\int_0^1 \hat{\varphi}_r(\xi)d\xi,$$

for $r \in \{0, 1, 2\}$ and $k \in \{2i, 2i+1, 2i+2\}$. This amounts to

$$b = \frac{h}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}.$$

For the boundary conditions, the reasoning is exactly the same as for linear elements.

We recall that for general FEM code the reasoning to build the stiffness matrix and load vector is as follows:

Set $A = 0$ and $F = 0$

Loop over the elements :

Build elementary stiffness matrix A_e on the element e

Build elementary load vector F_e on the element e

Find the global location of the element and add A_e, F_e in the right position

In our setting since we have a PDE with constant coefficients, the elementary stiffness and load are constant and this is why we can identity B and b as in the exercise.