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TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Fall 2022

Exercise set 1

1 Say if the following statements are **true** or **false**, and give a motivation.

a) The set $\mathcal{S} = \{v \in C^0((0,1)) : v(\frac{1}{3}) + v(\frac{2}{3}) = 1\}$ is a linear vector space.

b) Given two functions $f, v \in L^2(0,1)$, then the product fv lives in $L^1(0,1)$.

c) Consider $b(u,v) = \int_\Omega a(x)u(x)v(x)dx$, $\Omega \subset \mathbb{R}^n$ compact, with $a : \Omega \to (0, +\infty)$ a given continuous function, and $u, v \in L^2(\Omega)$. Then there is an $M > 0$ such that $b(u,u) \geq M\|u\|^2_{L^2}$ for any $u \in L^2(\Omega)$, i.e. b is coercive.

d) The only $w \in H^1(\Omega)$ for which $|w|_{H^1(\Omega)} = 0$ (the H^1 semi-norm), is $w = 0$. Would you answer the same even for $w \in H^1_0(\Omega)$?

2 Consider this model problem with variable coefficients:

$$\begin{aligned} -(a(x)u'(x))' &= f(x), \quad x \in I = [0, L], \\ a(0)u'(0) &= k_0(u(0) - g_0) \\ -a(L)u'(L) &= k_L(u(L) - g_L) \end{aligned}$$

where $a > 0$, f are given continuous functions, and $k_0 \geq 0$, $k_L \geq 0$, g_0, g_L are given parameters.[1]

a) Derive the weak formulation $b(u,v) = F(v)$, $u, v \in V$ of the problem, also specifying the functional space V where it is defined.

b) Is the bilinear form $b(\cdot,\cdot)$ symmetric?

c) Is the bilinear form $b(\cdot,\cdot)$ continuous?

d) Is the linear functional $F(\cdot)$ continuous?

3 **a)** For which $\alpha \in \mathbb{R}$ does the function $f(x) = |x|^\alpha$ live in $L^2([-1,1])$? When in $L^2([1,\infty))$? When in $L^2(B_1(0))$ with $B_1(0) = \{x \in \mathbb{R}^2;\ |x| < 1\}$?

b) Let $D \subset \mathbb{R}$ be a closed, bounded subset of $\mathbb{R}$ and $f \in C^0(D)$. Show that $f \in L^2(D)$. Does it live in $H^1(D)$ too?

[1] For the moment we do not go into why some of these functions/parameters need to be positive, but this is necessary for existence and uniqueness of the solution.

c) Let $\Omega \subset \mathbb{R}$ be some open interval. Recall that a weak derivative of $u : \Omega \to \mathbb{R}$ is a function $v : \Omega \to \mathbb{R}$ such that

$$\int_\Omega u(x)\phi'(x)dx = -\int_\Omega v(x)\phi(x)dx$$

for any smooth and compactly supported function $\phi \in C_c^\infty(\Omega)$. Show that the weak derivative, when it exists, is unique. Show that when $u \in C^1(\Omega)$, then the weak derivative coincides with $u'(x)$, i.e. its strong derivative.

d) Let

$$f_1(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ x - 1 & \text{if } 1 < x < 2 \\ 1 & \text{if } 2 \leq x < 3 \end{cases}$$

$$f_2(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 & \text{if } 1 \leq x < 3 \end{cases}$$

for $x \in \Omega := (0, 3)$. Show that $f_1, f_2 \in L^2(\Omega)$. Show that $f_1 \in H^1(\Omega)$ finding its weak derivative. Show then $f_1 \notin H^2(\Omega)$ and $f_2 \notin H^1(\Omega)$.

4 We consider the Dirichlet problem

$$\begin{aligned} -u''(x) &= 1, \quad x \in [0, 1] \\ u(0) &= 0 \\ u(1) &= 1. \end{aligned}$$

Let $\mathcal{T}_h$ be a triangulation of $[0, 1]$, i.e. a set of points $0 = x_0 < x_1 < ... < x_N = 1$. Let

$$X_h^1 = \{v \in C^0([0, 1]) : v|_{[x_i, x_{i+1}]} \in \mathcal{P}_1 \ \forall x_i, x_{i+1} \in \mathcal{T}_h\},$$
$$X_h^2 = \{v \in C^0([0, 1]) : v|_{[x_i, x_{i+1}]} \in \mathcal{P}_2 \ \forall x_i, x_{i+1} \in \mathcal{T}_h\}$$

be the two finite-dimensional spaces on which we will approximate the solution. The corresponding Galerkin problem is then

$$\begin{aligned} a(u_h, v_h) &= F(v_h), \quad \forall v_h \in X_h^i, \ i = 1, 2 \\ a(u, v) &= \int_0^1 u'(x)v'(x)dx \\ F(v) &= \int_0^1 v(x)dx. \end{aligned}$$

We now start working with the space of piecewise linear polynomials X_h^1. In this case, the problem can be solved by the following Python code, where $\mathcal{T}_h$ is the set of 20 equidistant points in $[0, 1]$

```
import numpy as np
import numpy.linalg

n = 20
x = np.linspace(0, 1, n)
```

```
A=np.zeros((n, n))
b = np.zeros(n)
h = 1/n

for i in range(n-1):
    A[i:i+2, i:i+2] = A[i:i+2, i:i+2] +
                      np.array([[1,-1], [-1, 1]])/h
    b[i:i+2] = b[i:i+2] + h/2
A[0,0:2] = [1 , 0] #so that (Au)[0] = u[0]
A[-1,-2:] = [0 , 1] #so that (Au)[N] = u[N]
b[0] = 0 #so that we get u[0] = 0
b[-1] = 1 #so that we get u[N] = 1
u = numpy.linalg.solve(A,b) #better with sparse solvers
```

a) Why are the increments for the stiffness matrix done with that 2×2 matrix?

b) Modify the code so that instead of Dirichlet boundary conditions, the problem solved has the following mixed boundary conditions

$$u(0) = 0, \;\; u'(1) = 1.$$

c) What is the exact solution to this problem? Plot your finite element solution and the exact solution in the same plot.

d) Modify your code to solve the problem using quadratic elements, i.e. where $V_h = X_h^2$ as defined above. Recall that the vertices defined for linear elements, need some additional nodes in order to define the FEM discretization. More precisely, each element will be characterized by three nodes $[x_i, x_{i+1}, x_{i+2}]$. The stiffness matrix A should be constructed from 3×3 sub-blocks of the form

$$\frac{1}{3h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

where h is the width of the element. Explain how this form is derived.