Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods
Fall 2021

Exercise set 2

$\boxed{1}$ The goal of this exercise is to recall the basic properties of Gaussian quadrature, so that we can then apply it to solve a PDE with FEM in the next exercise.

a) Consider the definite integral

$$\int_a^b f(x)dx$$

with $b - a = h$. What is the expression for the Gaussian approximation of the integral if you work with m quadrature points? In this and the next points, use as a reference interval where the nodes live $[0, 1]$, instead of the usual one $[-1, 1]$.

b) Since in the FEM formulation we need even to compute integrals involving derivatives, we are now interested in applying Gaussian quadrature to approximate

$$\int_a^b f'(x)g'(x)dx.$$

What is the expression, with m quadrature nodes, of the Gaussian approximation of this integral? Work again with the reference interval $[0, 1]$.

We focus on the mass matrix having entries

$$M_{ij} = \int_a^b \varphi_i(x)\varphi_j(x)dx$$

and on the stiffness matrix whose entries are

$$K_{ij} = \int_a^b \varphi_i'(x)\varphi_j'(x)dx$$

c) Let us now consider the interval $[0, 1]$, i.e. $h = 1$. Define on this interval the two linear basis functions

$$\hat{\varphi}_0(\xi) = 1 - \xi, \quad \hat{\varphi}_1(\xi) = \xi.$$

Compute the element mass and stiffness matrices for this interval.

d) Repeat the same reasoning for quadratic basis functions, again on the reference element $[0, 1]$. What are the expressions of the element stiffness and mass matrices in this setting? Recall that the reference basis functions are now

$$\hat{\varphi}_0(\xi) = (\xi - 1)(2\xi - 1) \quad , \quad \hat{\varphi}_1(\xi) = 4(1 - \xi)\xi \quad , \quad \hat{\varphi}_2(\xi) = \xi(2\xi - 1)$$

e) Write a Python/Matlab code which verifies that the calculated stiffness and mass matrices for quadratic basis functions are correct. Even if not necessary, compute the involved integrals with Gaussian quadrature with 5 nodes (this will be even used in the last exercise).

2 (Barycentric coordinates) Consider the triangle $\mathcal{T} \subset \mathbb{R}^2$ of vertices $A = (1, 1)$, $B = (1, 0)$, and $C = (-1, 1)$.

a) Compute the barycentric coordinates of the points $(1/2, 1/2)$ and $(-1/2, 1/2)$.

b) How can we easily verify if a point $Q \in \mathbb{R}^2$ is inside or outside $\mathcal{T}$ using barycentric coordinates?

c) Consider the integral

$$\int_{\mathcal{T}} f(x, y)\, dxdy, \quad f : \mathbb{R}^2 \to \mathbb{R}.$$

How can you rewrite it in terms of barycentric coordinates?

d) Transform the integral

$$\int_{\mathcal{T}} (x + y)\, dxdy$$

in barycentric coordinates and compute it.

3 Consider the boundary value problem

$$-u_{xx} + \sigma u = f \quad , \quad u(a) = u_1, u(b) = u_2.$$

We want to get an approximation of the solution u both with linear and quadratic elements. Moreover, we want to get measurements of the L^2 error.

a) First of all, write down the Galerkin formulation of the problem, defining the involved functional spaces.

b) Construct then a finite element code which can solve this BVP and requires as input parameters just

- p, which is the degree of the basis functions,
- N, which is the number of uniform elements of the triangulation,
- σ, a, b, u_1, u_2, which are the parameters of the BVP.

To assemble the load vector f, use Gaussian quadrature with 5 points. Recall that the elements are C^0-continuous, and this will influence the overlapping between elements in the global mass and stiffness matrices M and K.

c) Compute the error in the continuous L^2 norm and create convergence plots. Recall that

$$E = \sqrt{\int_a^b |u(x_i) - u_h(x_i)|^2},$$

where $u_h(x) = \sum_{i=0}^{n} \varphi_i(x)u_i$. The convergence plots, should be done both for $p = 1, 2$, and should consider $N \in \{4, 8, 16, 32, 64, 128, 256\}$. Repeat these tests for the two following problems

- For the first simulation, let $\sigma = 0$, $a = 0$, $b = 2$, $u_1 = 0$, $u_2 = 0$, $u(x) = \sin(\pi x)$ and $f(x) = \pi^2 \sin(\pi x)$.
- For the second simulation, let $\sigma = 2$, $a = -2$, $b = 2$, $u_1 = e^{-4}$, $u_2 = e^{-4}$, $u(x) = e^{-x^2}$, and $f(x) = 4(1 - x^2)e^{-x^2}$.